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A COURSE

OF

NATURAL PHILOSOPHY.

A COURSE

OF

NATURAL PHILOSOPHY,

CONTAINING

THE ELEMENTS OF

MECHANICS, HYDROSTATICS, AND OPTICS,

FOR

SCHOOLS, COLLEGES,

AND

CANDIDATES FOR MATRICULATION AT THE
UNIVERSITY OF LONDON.

BODLEIANA · BIBLIOTH.

BY

RICHARD WORMELL, M.A., B.Sc.,
Author of Mechanics, Hydrostatics, &c.

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PREFACE.

THE present work is intended chiefly for students preparing for Matriculation at the University of London. It is introductory to the Author's works for the Examinations subsequent to Matriculation, and follows the same form and order; so that students in pursuing the subjects further will be able easily to connect what they have to learn with what they have already learnt. At the same time the book by no means represents the narrowest view which may be taken of the Curriculum of the London University, but will, it is hoped, be found useful in schools generally, as containing a systematic explanation of the more elementary principles of this branch of Natural Philosophy, and being calculated to lay a sound foundation for a more advanced study of the subject.

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MECHANICS.

I.—INTRODUCTION.

1. A BODY is *at rest* when it constantly keeps the same position in space, and it is *in motion* when it occupies successively different positions.

2. To illustrate these conditions, let us place a stone or other body on a horizontal table. If left to itself, the stone will remain for an indefinite length of time in the same position.

Imagine the table to be suddenly removed; the stone will no longer remain at rest, but will fall to the earth. Why does it take that particular direction? A physicist would tell us that there is a constant attraction between the earth and the stone, and that the movement of the stone is due to this cause.

Whenever a body at rest is made to move, there always exists a cause of the motion, the action of which is due to the presence, near or remote, of another body; in other words, a body without life cannot move of itself.

3. If the stone be projected along a level road, the speed with which it leaves the hand will not be maintained, but will be gradually diminished, until finally

the stone will stop in its course. If, instead of the road, the frozen surface of a lake be chosen, the same stone thrown with the same force will travel much further on the ice than on the road. And if, instead of the irregular stone, we roll a smooth ball of ivory on the ice, the distance traversed will be greater still. It is evident, therefore, that the stone is gradually stopped by the resistances it encounters.

Similarly, whenever a body ceases to move, it does so because its motion is destroyed by the resistances it meets with. The more we diminish these resistances, the longer and the further will the body move; and, consequently, if we imagine that they are all suppressed, we shall be led to the conclusion that the body under these circumstances would continue to move for an indefinite length of time; in other words, that a body cannot of itself alter its speed.

4. Neither can it change the direction of its motion.

If no obstacle be encountered in its course, the ivory ball thrown on the ice will turn neither to the right nor to the left. It is true that a stone thrown into the air returns to the ground, but this is because its weight tends constantly to bring it to the earth. Conceive the weight and the resistance of the air removed, and the stone will continue to move in a straight line with uniform speed. This fundamental principle in Mechanics is termed the First Law of Motion, and may be enunciated thus:—

When a body is not acted on by any external agent, if it be in a state of rest it will remain so, and if it be in motion it will continue to move in a straight line with uniform speed.

Force.

5. When we raise a burden, set in motion a body at rest, or arrest a body already in motion, we are conscious of exerting a certain effort, and the term force, which we give this effort, is naturally applied to similar causes.

A force is therefore any cause tending to move a body to change its motion, or to keep it at rest when other forces are acting upon it.

For a long time the forces employed by man were only those furnished by human and animal labour; but, in proportion as progress in science has been made, we have not only utilised these natural mechanical forces, but have subjected to our use others which were before unknown; as, for example, forces arising from the motion of the air, a running stream, the changing of water into vapour by heat, chemical action, electricity, &c.

6. Forces from different sources, however, may be compared with one another, and are capable of numerical valuation.

The science of Mechanics does not consider the nature of the forces, but treats only of the properties common to all; and a force will be regarded as fully determined when we know the point at which it is applied, its direction, and its intensity.

7. Forces do not always have the effect of producing or modifying the motion of a body; other forces acting at the same time may counteract them. A weight held in the hand is not the less heavy because it does not fall; it is prevented from falling by a force exactly

equal to it, but opposite in direction. If the weight be placed on the table, it is still prevented from falling by a force exerted by the table. This force is termed a *reaction*, or a *resistance*. Whenever a body acted on by a force is at rest, we may at once conclude that there exists a resistance which counteracts the force. When two or more forces neutralise one another, so as to keep the body or the particle on which they act at rest, they are said to be *in equilibrium*.

8. The science of Mechanics may be conveniently divided into two parts.

Statics treats of forces in Equilibrium.

Dynamics treats of forces which produce motion.

II.—STATICS.

9. Three things have to be considered in a force.

1st. Its intensity or magnitude.
2nd. The point of application.
3rd. The direction.

The Magnitude of a Force.

10. Two forces are equal when, if applied at the same point in opposite directions, they will be in equilibrium. One force is twice another when the first will counteract two forces, each equal to the second, applied to the same point, but in the opposite direction; so also one force is thrice another when three of the latter

will be required to keep the former in equilibrium, and so on.

11. The magnitude of a force is estimated by stating how many times a given unit will be equal to it. The unit usually employed is a pound. Although forces are obtained from different sources, they may all be compared with the pound weight.

We will describe an instrument which may be used to compare forces of different kinds with the unit of weight.

12. Figure 1 represents a spring balance for this purpose, consisting of an elastic band of steel, B O A, to the ends of which metallic graduated arcs are attached. The outer arc, D A, is fixed to the lower arm, passes through an aperture in the upper, and terminates in a ring, F. The inner arc is attached to the upper arm, passes freely through the lower, and terminates in a hook, G. Place the ring on a firm support, and hang a weight of one pound upon the hook. The two branches of the steel band will approach each other until the elasticity of the bar counterbalances the weight. Mark on the arc, A D, the position of the bar. Suspend in a similar manner weights of 1 lb., 2 lbs., 3 lbs., &c., and make the corresponding marks until the

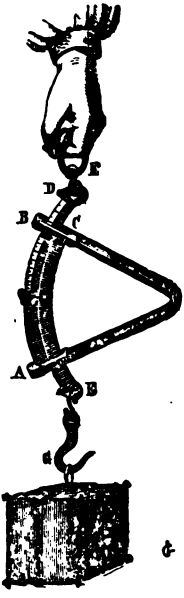


Fig. 1.

graduation of the arc is completed. Any force which when applied to the instrument produces a certain deflection is equal to the weight which would produce the same.

A spring balance used to compare forces differing in kind may be termed a *dynameter*.

Figures 2 and 3 represent other forms of the instrument. If the ends of the bar, A O B in Fig. 1 were united to a similar bar, by joints at A and B, the strength of the instrument would be greatly increased, and by means of such an arrangement we should be able to compare even forces of great magnitude with the unit of weight. If we wish, for example, to measure the force with which a horse draws a carriage along a paved road, it will only be necessary to interpose the dynameter between the horse and carriage, attaching one to the ring, and the other to the hook.

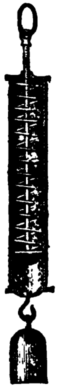


Fig. 2.

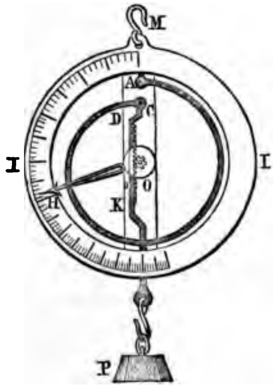


Fig. 3.

The Direction of a Force.

13. When a body is suspended by a thread (Fig. 4), the thread takes a determined direction, termed the vertical; if it be suddenly cut, the body will fall in the direction of the vertical. This is therefore termed the direction of the weight.

The direction of a force applied to a point is the straight line along which the point would be displaced if it were free to move.

Fig. 4.

Graphic Representation of Forces.

14. In the three properties we have described, forces may be represented by straight lines. For example, the direction of the line may represent the direction of the force, the extremity of the line the point of application of the force, and if we agree to represent a unit of weight by a unit of length, as for instance, by taking a line of three inches to represent a force of three lbs., then the magnitude of the line will represent the magnitude of the force.

The Transmissibility of Forces.

15. If a weight be attached by a cord to the spring balance in Fig. 1, the effect will be the same at whatever point in the cord the weight be tied. Similarly, a force may be applied to a body directly, or by the interposition of a rigid rod, and, supposing the rod to be supported independently, the result will be the

same. The general principle here illustrated may be stated thus. *A force may be applied at any point in the line of its direction, provided this point be connected with the first point of application by a rigid and inextensible straight line.*

Let A B (Fig. 5) be a string, at the extremities of which equal forces are acting in opposite directions, it is clear that the string will be in equilibrium. Take any other point, C, in the string, and remove the force, F, from A to C, still the force will be in equilibrium. Hence we may consider the force applied at one end to be transmitted throughout the string, and we may suppose two opposite forces at any point, each equal to F. Either of these forces is termed the *tension* of the string. Suppose the string to pass round a smooth peg, ring or surface, in this case also *the tension of the string is the same at every point.*

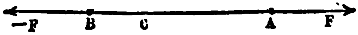


Fig. 5.

Resultant of Forces in the same Straight Line.

16. If we hang two weights of 1 lb. each to the dynameter, the indication of the instrument will be precisely the same as if a single 2 lb. weight were attached to it. Three weights of 1 lb. each produce the same effect as a single 3 lb. weight. Similarly, forces of 3 lbs. and 5 lbs. may be replaced by a single force of 8 lbs. And, generally, it is evident that *when two or more forces act upon a point in the same straight line*

in the same direction, their effect will be equivalent to that of a single force equal to their sum.

Conversely, if for a single force two others, the sum of which is equal to the first, be substituted, the effect will be the same.

Let us now apply at the point o two unequal forces in opposite directions, the one, F′, acting horizontally from left to right, and the other, F, horizontally from right to left (Fig. 6). If F′ be a force of 2 lbs. and F of 3 lbs., we may substitute three single forces of 1 lb. each for the latter, and two of them will then be in equilibrium with the former. There remains a free force of 1 lb. acting from right to left. The body under the influence of the two forces is therefore in the same condition as if it were acted on by a force, F—F′, in the direction of the greater. Whenever two unequal forces act on a particle in the same straight line, but in opposite directions, the greater may be divided into two parts, one of which is equal to the less, and will therefore neutralise it, and an effective force will remain in the direction of the greater, equal to the difference of the forces.

Fig. 6.

The single force which will produce the same effect as several forces acting together, is termed the resultant of the forces; hence the general proposition explained above may be enunciated thus.

When any number of forces act upon the same point, in the same straight line, their resultant may be found by taking the difference between the sum of all the forces acting in one direction, and the sum of all the forces acting in the other, the direction of the resultant being that of the greater sum.

Let us agree to write a + sign before all the forces acting in one direction, and a — sign before all those acting in the opposite direction, and then we may state the rule thus:—*The resultant of a number of forces acting in the same straight line is equal to the algebraical sum of the forces.*

17. The forces which are thus combined are termed the *Components*, and the process of finding the resultant is termed the *Composition of Forces.* The converse operation, namely, that of finding Component forces which shall have a given *resultant*, is termed the *Resolution of Forces.*

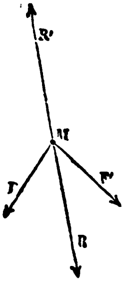


Fig. 7.

The resultant is the single force which will produce the same effect as the components; hence, if to a system of forces a new force be added, exactly equal to the resultant, and opposite in direction, this force will keep the system in equilibrium (Fig. 7). Thus the problem to find the resultant also gives the force required to keep the system in equilibrium.

EXERCISES.

1. If a force which will support a weight of 18 lbs. be represented by a line 3 ft. long, how long must be the line which will represent a weight of 1½ cwts.?—*Ans.* 28 ft.

2. In the above case, what weight will a line of 17 in. represent?—*Ans.* 8½ lbs.

3. The force required to balance two others acting together in the same straight line is five times one-fourth of larger force, what multiple is it of the smaller?—*Ans.* 5.

4. When a ton is represented by a line 5 ft. 4 in. long, what force will a line 7 in. represent?—*Ans.* 2 cwt. 21 lbs.

5. The weight of a cubic foot of a substance is 6 cwt., and is represented by a line a foot long, what will be the length of the line required to represent 720 cubic inches of the same substance? —*Ans.* 5 inches.

6. When two forces act together they have a resultant of 12 lbs., and when they act in opposite directions their resultant is 2 lbs.; find the forces.—*Ans.* 5 lbs. and 7 lbs.

7. A string supports a weight of 4 lbs. at its extremity, another weight of 5 lbs. above the first, and a third of 7 lbs. above the second: find the tensions of the three parts of the string. If the tension of the middle part be represented by a line 11¼ in. long, what will be the length of the lines required to represent the others?—*Ans.* Tensions, 4 lbs., 9 lbs., 16 lbs.; lines, 5 in. and 20 in.

III.—THE COMPOSITION AND RESOLUTION OF FORCES.

Forces in the same Plane meeting at a Point.

18. Suppose two forces not in the same straight line act on a point M (Fig. 8). If left to the action of these forces only, the point will begin to move in a certain direction. There must, therefore, be a single force which will produce the same effect, and therefore, also, a force which will counteract this effect. We have to consider how we can find the magnitude and direction of this force.

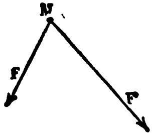


Fig. 8.

Over two pulleys, M and N (Fig. 9), capable of moving without friction, pass a fine thread of silk. To the middle point B tie a very light strip of wood E D. Take a number of small equal weights, and attach two of them to each end of the thread, and three to the extremity E of the strip E D. The point B will descend to a definite position, and will then remain at rest.

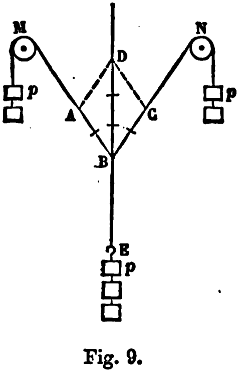


Fig. 9.

Let us examine the position of equilibrium. First, we shall find that the rod E D will bisect the angle *M B N; and* this will always be the case whatever the weights, provided that those at M and N are equal. Again, take a certain length—an inch, for example—to represent the weight p, and measure off on B N and B M as many inches as there are weights at each end of the thread and through the points A and C thus determined, draw A D and C D parallel to B C and A B respectively. These lines will meet the rod in the same point D, and it will be found that the line B D will contain as many inches as there are weights attached to E.

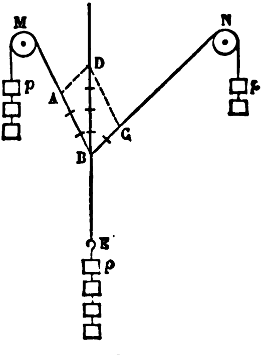


Fig. 10.

Instead of equal weights suspend 3 p at one end of the thread, 2 p at the other, and 4 p to the rod (Fig. 10). The form of the figure will be changed, but if we measure on B M three units of length, and draw A D parallel to B C, also measure on B C two units, and draw C D parallel to B A, still we shall find that these straight lines intersect at a point D on the rod, and that the line B D contains as many units of length as there are weights at E.

This experiment illustrates the fact, that, in order to find the resultant of two forces acting on a point, we must take on their directions lengths proportional to the forces, and complete on them a parallelogram. The diagonal of the parallelogram through the point of application will represent the resultant of the forces in magnitude and direction.

19. We may here remark that the thread must be tied to the rod, in order that there may be equilibrium with unequal weights at M and N. If the weight at B be attached by a ring, as in Fig. 11, the tension in

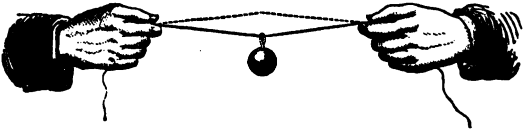


Fig. 11.

the two parts of the string must be equal. The parallelogram in this case is equilateral.

The proposition above explained is termed the Parallelogram of Forces.

20. *If two forces acting on a particle be represented in magnitude and direction by two adjacent sides of a parallelogram, the resultant of these forces will be represented in magnitude and direction by that diagonal of the parallelogram which passes through the particle.*

From this important law we may make several deductions.

(i.) If two equal forces F, F′, act on a point M, the resultant bisects the angle between them (Fig. 12).

(ii.) Three equal forces making with each other angles of 120° (Fig. 13) will be in equilibrium, for the resultant of any two will be equal and opposite to the third.

(iii.) We may always replace a single force by two others which will have the same effect as the single force, and which will act on the same point in any given directions.

(iv.) When the two forces act at right angles, the *resultant will be found* by adding the squares of the numbers, measuring the forces, and taking the square root of the sum.

Thus, if P and Q be the forces acting on the point at right angles to one another, and R their resultant, then $R^2 = P^2 + Q^2$.

EXAMPLES.

1. Find the resultant of two forces of 8 lbs. and 15 lbs. respectively acting on a particle at right angles to one another.

$$\begin{aligned} P^2 &= 8^2 = 64 \\ Q^2 &= 15^2 = 225 \\ \hline R^2 &= 289 \\ \therefore R &= 17 \text{ lbs.} \end{aligned}$$

2. If the resultant of two forces acting at right angles be 53 lbs. and one of the forces be 45 lbs., find the other force.

$$\begin{aligned} R^2 &= 53^2 = 2809 \\ P^2 &= 45^2 = 2025 \\ \hline \text{Subtract} \therefore Q^2 &= 784 \\ \therefore Q &= 28 \end{aligned}$$

21. Remark, that the unit of length taken to represent a unit of force may be anything we please, but when a certain line has been taken to represent a given force, the unit is fixed, and must be maintained throughout the problem.

Example 3.—A boat is held at rest in a stream by cords, A O, B O (Fig. 14), attached to stakes in the opposite banks, and a rope, C O, fastened to the boat C. Show how to find the ratio *of the tensions in* the cords.

Take *any* length O A′ to represent the tension in O A. Through A′ draw a line parallel to O B, and produce C O to meet it. From the point of intersection draw a parallel to O A, meeting O B in B′. The ratio of O A′ to O B′ will be the ratio of the tensions. The parallelogram might have been constructed on any two of the lines. If the absolute value of one of the tensions is known, as, for example, that in A O = 50 lbs., the unit taken $\left(=\frac{A'O}{50}\right)$ is known, and the tension in the other cords may now be found.

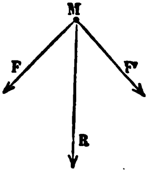


Fig. 12.

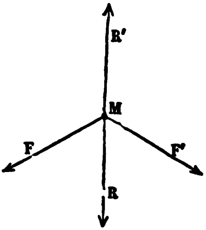


Fig. 13.

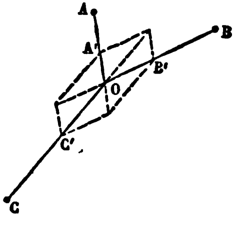


Fig. 14.

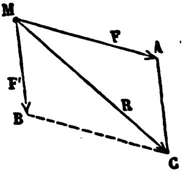


Fig. 15.

4. Show how to find the magnitudes of the forces when the magnitude and direction of the resultant are given, and the directions of the forces.

Let M represent the point on which the forces act (Fig. 15), *draw the* lines M A, M B, having respectively the directions of *the two forces, and the* line M C, having the direction of the resultant. Take some unit of length to represent a unit of weight, and on the scale chosen cut off M C to represent the magnitude of the resultant R. Through C draw C A parallel to M B, and C B parallel to M A, then M A, M B, will represent the forces in magnitude, for the forces represented by these lines will by the parallelogram of forces have for resultant the given force R.

5. Two forces which act at right angles on a particle have the ratio of 9 to 40, and their resultant is 123 lbs.; find the magnitudes of the forces.

Since the forces are as 9 to 40, one is nine times some unit of which the other is 40 times.

$$\text{Let } P = 9a \qquad \therefore Q = 40a$$

$$P^2 = 81a^2$$

$$\underline{Q^2 = 1600a^2}$$

$$R^2 = 1681a^2 = (41a)^2$$

$$\text{but } R = 123 \text{ lbs.}$$

$$\therefore 41a = 123 \text{ lbs.}$$

$$a = 3 \text{ lbs.}$$

$$\therefore P = 27 \text{ lbs. and } Q = 120 \text{ lbs.}$$

EXERCISES.

1. Define the term *force*, and show that all forces may be compared with *weight*.

2. In what respects may a force be represented by a straight line?

3. What is the Principle of the Transmissibility of Force?

4. Define the term *Resultant*, and show how to find the resultant of two or more forces acting in the same straight line.

5. Enunciate the Parallelogram of Forces.

6. Show that if the directions of two forces acting on a point include an angle, their resultant is less than if they act in the same straight line and in the same direction.

7. What must be the directions of two forces that their resultant may be the least possible?

8. Three ropes, P A, Q A, R A, are knotted together at the point A; *P A is attached to* a tree, Q A and R A are pulled by

two men; having given the angle Q A R, and the force exerted by each man, show how to find the pressure on the tree.

9. Show how to find the resultant of two forces when the directions of the two forces, the direction of the resultant, and the magnitude of one force are given.

10. Show how to find the direction of the resultant of two forces when the magnitudes of one force and the resultant as well as the directions of the two forces are given.

11. Show how to find the magnitude of two forces, having given their directions and the direction and magnitude of their resultant.

12. Two forces, P and Q, act on a point at right angles; P = 16 lbs., Q = 63 lbs.; find R.—*Ans.* 65 lbs.

13. If P = 13 and Q = 84, the angle being right, find R.—*Ans.* 85.

14. The angle between P and Q is right, P = 112 cwt. and R = 113; find Q.—*Ans.* 15 cwt.

15. The ratio of P to Q is 33 : 56, and the resultant is 130 lbs.; find P and Q, the angle PQ being right.—*Ans.* 66 lbs. and 112 lbs.

16. The resultant being 85 lbs. and Q 77 lbs., find P when ∠ PQ is right.—*Ans.* 36 lbs.

17. P, the greater of two forces, is 3·2 lbs. less than R, which is 30½ lbs.; find Q, the less, PQ being a right angle.—*Ans.* 13·6 lbs.

18. The smaller of two forces which act at right angles is 7·2 lbs., and the sum of the resultant and the larger force is 259·2 lbs.; find the resultant and the larger force.—*Ans.* 129·7 and 129·5.

19. For a given vertical force a horizontal and an oblique force have to be substituted; the horizontal force equals the given force; find the magnitude and direction of the oblique force.

20. Two forces which act at right angles are to one another as 16 to 63, and the resultant is 13 lbs.; find the forces.—*Ans. 3·2 and* 12·6.

IV.—APPLICATIONS OF THE PARALLELOGRAM OF FORCES.

22. By means of the Parallelogram of Forces we may always find the resultant of two forces acting on a point when we know the magnitudes of the forces and the angle between them, but the magnitude of the resultant can be calculated numerically without the aid of trigonometry only in the case of a few angles, namely, when the given angle is one of the angles of the following triangles:—

(i.) A right-angled triangle in which the acute angles are respectively 30° and 60°. Such a triangle is half an equilateral triangle, and it is easily proved that

The shortest side = half the longest.

The third side = $\sqrt{3}$ × the shortest.

(ii.) An isosceles right-angled triangle (in which of course the angles are 45°, 45°, and 90°.)

The hypothenuse = either side × $\sqrt{2}$.

23. To find the diagonal of the parallelogram when the two sides and included angle are given.

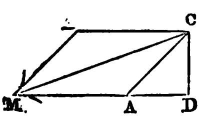


Fig. 16.

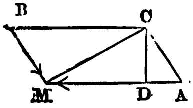


Fig. 17.

Let A M B C be the parallelogram, and let C D be the perpendicular on M A from the point C.

First, let the angle at M be acute, as in Fig. 16, then

$$
\begin{aligned}
&& \mathrm{M\,C}^2 &= \mathrm{M\,D}^2 + \mathrm{D\,C}^2 \\
&\text{or since} & \mathrm{M\,D} &= \mathrm{M\,A} + \mathrm{A\,D} \\
&& \mathrm{M\,C}^2 &= (\mathrm{M\,A} + \mathrm{A\,D})^2 + \mathrm{D\,C}^2 \\
&& &= \mathrm{M\,A}^2 + \mathrm{A\,C}^2 + 2\ \mathrm{M\,A}\cdot\mathrm{A\,D}.
\end{aligned}
$$

Secondly, let the angle at M be obtuse, as in Fig. 17, then

$$
\begin{aligned}
\mathrm{M\,C}^2 &= \mathrm{M\,D}^2 + \mathrm{D\,C}^2 \\
&= (\mathrm{M\,A} - \mathrm{A\,D})^2 + \mathrm{D\,C}^2 \\
&= \mathrm{M\,A}^2 + \mathrm{A\,C}^2 - 2\ \mathrm{M\,A}\cdot\mathrm{A\,D}.
\end{aligned}
$$

Now it has been shown above that A D can be easily found from A C when the angle at M is either 60°, 30°, 45°, (180°—60°), (180°—30°), or (180°—45°).

Thus, if $\mathrm{CAD} = 60^\circ$, $\mathrm{DA} = \frac{1}{2}\,\mathrm{A\,C}$, and $\mathrm{CD} = \frac{\sqrt{3}}{2}\,\mathrm{AC} = \cdot 866\,\mathrm{A\,C}$.

if $\mathrm{CAD} = 30^\circ$, $\mathrm{DA} = \frac{\sqrt{3}}{2}\,\mathrm{AC}$, and $\mathrm{CD} = \frac{1}{2}\,\mathrm{AC}$.

if $\mathrm{CAD} = 45^\circ$, $\mathrm{DA} = \mathrm{DC} = \frac{\mathrm{AC}}{\sqrt{2}} = \frac{\sqrt{2}\,\mathrm{AC}}{2} = \cdot 707\,\mathrm{AC}$.

24. To find the resultant of two forces P and Q, when they act at one of the simple angles.

Let M B represent Q, ∴ A C also represents Q, and let A M represent P.

Then, if $\angle\,\mathrm{BMA} = 60^\circ$, $\mathrm{R}^2 = \mathrm{P}^2 + \mathrm{Q}^2 + \mathrm{PQ}$.
if $\angle\,\mathrm{BMA} = 30^\circ$, $\mathrm{R}^2 = \mathrm{P}^2 + \mathrm{Q}^2 + \sqrt{3}\cdot\mathrm{PQ}$.
if $\angle\,\mathrm{BMA} = 45^\circ$, $\mathrm{R}^2 = \mathrm{P}^2 + \mathrm{Q}^2 + \sqrt{2}\cdot\mathrm{PQ}$.
if $\angle\,\mathrm{BMA} = 120^\circ$, $\mathrm{R}^2 = \mathrm{P}^2 + \mathrm{Q}^2 - \mathrm{PQ}$.
if $\angle\,\mathrm{BMA} = 150^\circ$, $\mathrm{R}^2 = \mathrm{P}^2 + \mathrm{Q}^2 - \sqrt{3}\cdot\mathrm{PQ}$.
if $\angle\,\mathrm{BMA} = 135^\circ$, $\mathrm{R}^2 = \mathrm{P}^2 + \mathrm{Q}^2 - \sqrt{2}\cdot\mathrm{PQ}$.

Example. Find the resultant of two forces of 12 lbs. *and 8 lbs. respectively*, acting at an angle of 60°.

In Fig. 16 let $MB = 12$, $MA = 8$, and $\angle BMA = 60°$, then $AC = 12$, and since $\angle DAC = 60°$, $AD = 6$, $\therefore DC = 6\sqrt{3}$.

And $MD = MA + AD = 8 + 6 = 14$

$$\therefore MC^2 = MD^2 + DC^2 = 14^2 + (6\sqrt{3})^2 = 196 + 108$$

$$\therefore R = \sqrt{(304)} = 17{\cdot}43558.$$

Or we may apply directly the formula—

$$MC^2 = MA^2 + AC^2 + 2\,MA \cdot AD$$

$$\therefore R^2 = 8^2 + 12^2 + 2 \times 8 \times 6 = 304.$$

Conditions of Equilibrium.

25. Any relations which must hold amongst the forces when there is equilibrium, and such that there is always equilibrium when the relations are fulfilled, are termed the Conditions of Equilibrium.

26. The condition of equilibrium of two forces acting on a point is that they be equal and opposite.

27. When three forces act upon a point, the condition of equilibrium is that any one of them shall be equal and opposite to the resultant of the other two.

28. When the three forces act at points in a solid body, and are not parallel, one condition of equilibrium is that their directions shall meet in the same point.

29. It is evident that we may add to or subtract from a system of forces any set of forces which are in equilibrium amongst themselves. For example, if we wish to find the resultant of three forces of 30 lbs., 50 lbs., and 60 lbs., acting at angles of 120°, since three forces of 30 lbs. acting at 120° will be in equilibrium, we may subtract 30 lbs. from each of the forces, and find the resultant of the remaining 20 lbs. and 30 lbs. *acting at an angle* of 120°.

EXERCISES.

1. Find the resultant of two equal forces of 15 lbs. acting on a point at an angle of 60°.

2. Find the resultant of equal forces of 20 lbs. acting on a point at an angle of 30°.

3. Find the resultant of equal forces of 40 kilogrammes, the angle between them being 45°.

4. Find the resultant of equal forces of P lbs., the angle between them being 120°.—*Ans.* P lbs.

5. Find the resultant of equal forces of 100 lbs., the angle between them being 135°.—*Ans.* 76·5.

6. Two forces of 10 lbs. and 42 lbs. act upon a point at an angle of 120°; find their resultant.—*Ans.* 38 lbs.

7. Forces of 8 and 12 lbs. act at an angle of 60°; find the resultant.—*Ans.* 17·43.

8. Forces of 9 and 11 lbs. act at an angle of 120°; find their resultant.—*Ans.* 10·14.

9. The resultant of two forces, P and Q, is perpendicular to P; show that it is less than Q.

10. Two forces, each equal to 50 lbs., act at an angle of 150°; find their resultant.—*Ans.* 25·8819.

11. At what angle must two equal forces act to produce the same effect as one of them?

12. Two rafters, making an angle of 60°, support a chandelier weighing 90 lbs.; what will be the pressure along each rafter? —*Ans.* $30 \cdot \sqrt{3}$ lbs., or 51·96 lbs.

13. For a given vertical force two other forces are to be substituted, one horizontal and the other making an angle of 45° with the vertical; find the magnitude of the forces.

14. The resultant of two equal forces acting upon a point at an angle of 90° is 10 lbs.; find the value of each component.—*Ans.* 7·07107 lbs.

15. The resultant of two equal forces acting upon a point at an angle of 135° is 10 lbs.; find the value of each component.—*Ans.* 13·0659 lbs.

16. The resultant of two forces acting upon a point at an angle *of 150° is 95 lbs.;* one of the components is 55 lbs.; find the *other component.—Ans.* $80\sqrt{3}$ lbs., or 138·56.

The Polygon of Forces.

30. When any number of forces F, F′, F″, F‴, act upon a point M, we may find the resultant of the whole by combining two, F and F′, by the Parallelogram of Forces (Fig. 18), then their resultant R′ and the third force F″, and so on. When this construction is made, it will be found that by neglecting the successive diagonals it amounts to the following rule.

31. From any point draw lines parallel and proportional to the forces; from the extremity of one of these lines draw a line parallel and equal to the next; from the end of this draw a line parallel and equal to the third, and so on. The line joining the open part of the polygon thus formed will be parallel and equal to the line representing the resultant. (Fig. 18.)

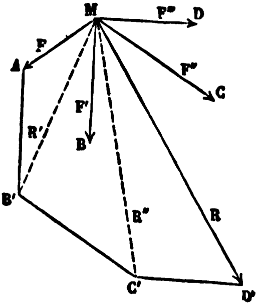


Fig. 18.

32. If the polygon be a closed polygon, the forces have no resultant, that is to say, there is equilibrium. For example, let M A, M B, M C, M D, represent forces acting on the point M. From A draw A B′, parallel and equal to M B; from B′ draw B′ C′, parallel and equal to M C; and from C′ draw C′ D′, parallel and equal to M D; then M D′ represents the resultant in magnitude and direction. If the polygon be closed, so that D′ coincides with *M, then the forces* are in equilibrium.

A particular case of this proposition is termed the triangle of forces.

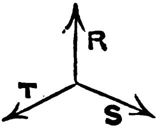


Fig. 19.

33. *When three forces acting on a particle can be represented in magnitude and direction by the three sides of a triangle taken in order, they will be in equilibrium.*

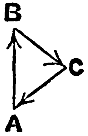


Fig. 20.

34. The converse of this proposition is also true. *When three forces acting on a particle are in equilibrium, the sides of any triangle which are parallel to the lines of action of the forces are also proportional to the forces.*

35. Let R S T be three forces acting on a point (Fig. 19), and let the sides of the triangle A B C be respectively parallel to the forces,

thus A B parallel to R
B C ,, to S
C A ,, to T

Then the first of the above propositions states that if AB : BC : CA as R : S : T, then the forces are in equilibrium; and the second states that if the forces are known to be in equilibrium, and the triangle A B C be constructed with sides parallel to the forces, then AB : BC : CA as R : S : T.

36. It must be remembered that these propositions apply only to forces acting *on the same point.*

Exercises.

1. Show how to find the resultant of a number of forces acting on a point.

2. Three equal forces act on a point at right angles; find their resultant.

3. Three forces, P, Q, and R, act on a point, the angles $\widehat{PQ}$ and $\widehat{QR}$ are each 60°; P = 2 lbs., Q = 8 lbs., R = 2 lbs., find the resultant.—*Ans.* 10 lbs.

4. Seven forces act on a point so that the angle between every consecutive two is 45°. The first force and the alternate forces from the first are each 5 lbs.; the second and alternate forces from the second are each 8 lbs., find the resultant of the whole. —*Ans.* 8 lbs.

5. If two lines, A B, C A, represent two forces acting on a point, the one towards the point and the other from it, show how to find the resultant.

6. Three forces of 119, 120 and 169 lbs. act on a point and keep it at rest; show that the angle between the first and second is a right angle.

7. A cord is attached to two fixed points, A and B, in the same horizontal line, and bears a ring weighing 10 lbs. at C, so that A C B is a right angle; find the tension in the cord.—*Ans.* $5\sqrt{2}$ lbs.

8. Two cords, A C = 44 inches, B C = 117 inches, are attached to points A and B in the same horizontal line and to a weight of 10 lbs. at C. The angle A C B is a right angle; find the pressures on A and B.—*Ans.* 3·52 and 9·36 lbs.

9. Prove that if two forces be represented by two diagonals of a parallelogram, their resultant will be represented by a line equal to twice one of the sides of the parallelogram.

V.—PARALLEL FORCES.

37. Take a bar of iron B C (Fig. 21), and attach it by the middle point D to a fine thread passed over a pulley E capable of moving without friction. To the other end of the cord attach a mass of lead and a hook A, which will exactly balance the bar. Take a number of equal weights and suspend one of them from each of two points on the bar equally distant from D, and two others from the hook; there will still be equilibrium. Take off the two weights from the bar and hang them one below the other at the point D: with this arrangement also there will be equilibrium. This is an illustration of the fact that two parallel forces, applied at two points in a body, produce the same effect as if they were applied together at the centre of the line which joins the points.

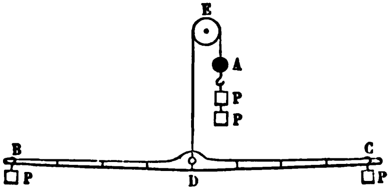


Fig. 21.

Replace one weight at B, suspend two others at C′, midway between C and D (Fig. 22), and hang three *from the* hook at A; in this case also there will be *equilibrium.*

If C'' be taken as the point of support, such that $DC'' = \frac{1}{4} DB$, then it will be necessary when one weight is placed at B to suspend four from C'' and five from A.

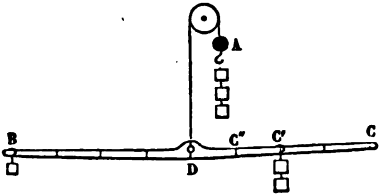


Fig. 22.

38. These experiments, which might be multiplied and varied indefinitely, show that two parallel forces, acting in the same direction, can be counteracted by a single force parallel to them; and consequently they may be replaced by such a force. Moreover, we learn that in magnitude the resultant is equal to the sum of the forces, and if one force be double or triple of the other, the point of application of the first must be twice or thrice nearer the resultant than that of the second; in other words, the distances of the forces from the resultant are inversely proportional to the forces.

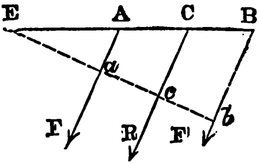


Fig. 23.

39. Let F, F′ be parallel forces acting towards the same direction at the extremities of a rod A B (Fig. 23).

Let R be their resultant and let C be its point of application. The principle explained above is expressed by the following equations:—

$$R = F + F'$$
$$F \cdot AC = F' \cdot BC$$

If E be any point in the continuation of BA then

$$F \cdot AE + F' \cdot BE = R \cdot CE$$

This is the general equation for determining the position of the resultant, that is to say, it includes all cases. For example, let E coincide with A, then

$$F \times 0 + F' \cdot AB = R \cdot AC \text{ or } F' \cdot AB = R \cdot AC$$

Let E coincide with B, then

$$F \cdot AB + F' \times 0 = R \cdot CB \text{ or } F \cdot AB = R \cdot CB$$

COMPOSITION OF PARALLEL FORCES ACTING TOWARDS OPPOSITE DIRECTIONS.

40. Let F and F′ be parallel forces, acting in opposite directions, and let F be the greater (Fig. 24). The force F may be decomposed into two forces; one represented by B B′ at B equal and opposite to F′, and the other equal to F — F′ at a point C, such that,

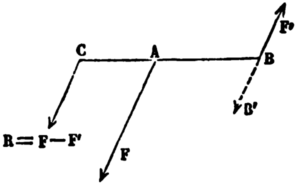


Fig. 24.

$$(F - F') \cdot CA = F' \cdot AB$$
$$\text{or } F \cdot AC = F' \cdot CB$$

The forces at B will be in equilibrium, and an effective force remains at C equal to F — F′. This, then, is the resultant R of the forces F and F′. Therefore the *resultant of two unlike* parallel forces acting on a solid body at points A and B is equal to their difference, and acts in the direction of the greater, through a point C in the line A B produced on the side of the greater force, such that

$$F : F' :: CB : CA$$

41. These cases may be collected under the following rule which applies to any number of parallel forces in the same plane.

To find the magnitude of the resultant take the sum of those forces which act towards one direction from the sum of those which act in the opposite direction.

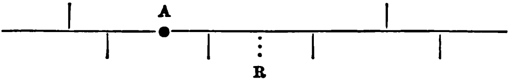


Fig. 25.

To find the position of the resultant draw a straight line cutting the directions of all the forces, take a point A on this line and find the distances of the forces from A measured along the line. Multiply each force by its distance from the fixed point A, before each product thus obtained write the proper sign and add all the products together.

Let x be the distance of the resultant R from A. Multiply R by x and write the result on one side of an equation, the sum of the preceding products being on the other.

42. We have yet to show how to determine the proper sign of each of the products. Imagine the line drawn from A to the force to be a rod jointed to the fixed point A, then if the force would tend to turn this rod in the direction of the hands of a clock make the *sign of the product* +, but if the force would tend to *turn the rod in the* opposite direction, then make the

sign —. Thus in Fig. 26 the product is + $F \cdot a$, in Fig. 27 it is — $F \cdot a$, in Fig. 28 — $F \cdot a$, and in Fig. 29 we have + $F \cdot a$.

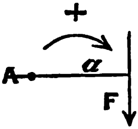


Fig. 26.

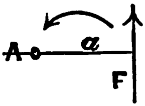


Fig. 27.

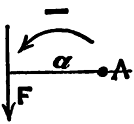


Fig. 28.

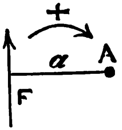


Fig. 29.

43. *Example 1.*—Parallel forces of 2 lbs. and 6 lbs. respectively act at the ends of a rod 16 inches long; find the magnitude and position of the force which will keep the rod at rest.

Let B C be the rod, and let D be the point of application of the force required, then D is also the point of application of the resultant of the given forces.

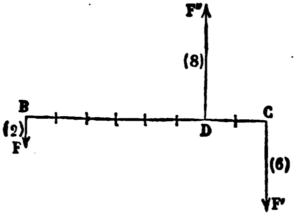


Fig. 30.

The magnitude of the resultant = 6 + 2 = 8. To find the position of the point D, we may proceed in several ways, thus:—

First. Since the arms must be proportional to the forces, that is to say, as 6 : 2, if the rod be divided into 8 equal parts, B D will be equal to 6 of these parts and C D to 2. Each part will be 16 ÷ 8, or 2 inches, therefore

$$BD = 12 \text{ inches and } CD = 4 \text{ inches.}$$

Second. Or we may let B D = x then C D = $(16 - x)$, but by the above rule

$$2\,BD - 6\,CD = 0$$
$$\text{or } 2x - 6\,(16 - x) = 0$$
$$\therefore x = 12$$

Thirdly. We may let B be the fixed point of reference and apply the above rule, letting $BD = x$

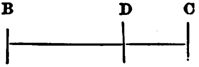


Fig. 31.

$$\text{then } F \times 0 + F' \times CB = R \times BD$$
$$\text{or } 2 \times 0 + 6 \times 16 = 8 \times x$$
$$\therefore x = 12$$

44. The above rule fails in the case of two equal and parallel forces which act towards opposite directions. Such a pair of forces has not a single resultant; the forces form what is termed a *couple.* A couple can be kept in equilibrium only by another couple in the same.

Centre of Parallel Forces.

45. To find the resultant of a number of parallel forces not in the same plane; as, for example, when the forces act at points in a solid body we must first find the resultant of two, then combine that resultant with a third, and so on.

Let F, F', F'', &c., be forces acting at points A, B, C, &c., in a solid body. The resultant R_1 of F and F' will act at a point L_1 in AB, such that $AL_1 : BL_1$ as $F' : F$. The resultant of R_1 and F'' will act at L_2 in the line L_1C, such that $L_1L_2 : CL_2$ as $F'' : R_1$. In this way we may proceed *until* we find

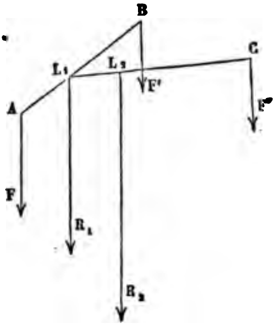


Fig. 32.

the point L at which acts the resultant R of all the forces.

46. Now suppose the direction of the forces to be changed, the forces being of the same magnitude, remaining parallel and being applied at the same points as before. The resultants R_1, R_2, &c., though altered in direction, will still pass through the points L_1, L_2, &c., and the final resultant R will pass through the point L. Hence *the resultant of a system of parallel forces acting at different points in a rigid body passes through a fixed point, the position of which is independent of the direction of the forces.* This point is termed the centre of the forces.

EXERCISES.

1. Show how to find the resultant of two given parallel forces acting towards the same direction, the distance between the forces being given.

2. Forces of 8 lbs. and 12 lbs. respectively act at points A and B 12 inches apart; find the magnitude and position of their resultant.

3. Show how to find the magnitude and position of the resultant of a number of parallel forces when the magnitudes and directions of the forces are given and their several distances from a fixed point.

4. What is a couple? How may a body be kept at rest when a couple is acting on it?

5. The smaller of two parallel forces of 24 and 30 lbs. respectively is 5 inches from the resultant; what is the distance between the larger force and the resultant?—*Ans.* 4 in.

6. The resultant of two forces is 66 lbs.; the smaller force is $\frac{10}{22}$ of the resultant and is distant from it 3 ft. 6 in.; find the other force and its distance from the resultant.—*Ans.* 36 lbs.; 2 ft. 11 in.

7. *The perpendicular* distance between two parallel forces

of 1 cwt. and 20 lbs. respectively is 2 ft. 9 in.; find the distance between the smaller force and the resultant.—*Ans.* 28 inches.

8. Two parallel forces which act at the extremities of a rod $5\frac{1}{3}$ ft. long have a resultant of 1 cwt.; one of the forces is 35 lbs; find the distances of the points of application of the forces from that of the resultant.—*Ans.* 20 and 44 inches.

9. A B is a rod acted on at A and B by parallel forces P and Q. C is the point of application of their resultant R. Given that R = 154 lbs., Q = 99 lbs., A C = $5\frac{1}{4}$ ft.; find A B.—*Ans.* 98 in.

10. In the last example, given that A B = 16 in., R = 104, and $\frac{P}{Q} = \frac{3}{5}$; find P, Q, A C, and B C.—*Ans.* 39 lbs., 65 lbs., 10 in., 6 in.

11. At points equally distant on a rod 20 inches long, weights of 1 lb., 2 lbs., 3 lbs., 4 lbs., and 5 lbs. are suspended. The rod is to be supported by a single thread; at what point must it be tied that the rod may remain horizontal?—*Ans.* $13\frac{1}{3}$ in. from the end.

12. A weight of $1\frac{1}{2}$ cwt. is carried by two men on a rod 8 ft. long. The weight is hung from the middle; one man is 1 ft. from one end and the other 2 ft. from the other end of the rod; find the weight borne by each.—*Ans.* ·9 cwt. and ·6 cwt.

VI.—MOMENTS.

47. Suppose a rod O D, capable of turning about the fixed point O, to be acted on by a force F, the tendency of the force to turn the rod about O depends on the magnitude of the force and on its distance from the point O. We might, for example, double this tendency, either by doubling the force, or by keeping the force the same and causing it to act at twice the distance from O. Hence the tendency of the force to

turn the rod about O is measured by the product of the force by the perpendicular O D.

48. *The product of any force* F *by the perpendicular from a point on its direction is termed the moment of the force with respect to the point.*

Thus if A M represent the force F (Fig. 33), the moment of F about O is A M × O D. This product is numerically *twice the area of the triangle having the line representing the force for base and the given point for apex.*

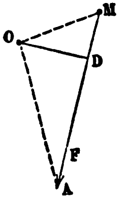


Fig. 33.

49. We show therefore that the moments of two forces are equal when we prove the equality of the triangles formed by joining the extremities of the lines representing the forces to the given point.

50. It is evident that the moment of a force about a point in its own direction is zero.

51. It is convenient to consider moments in one direction, as, for example, that of the hands of a clock as positive, and moments in the opposite direction as negative.

The moment of a force is therefore a product, the sign of which is determined in the way described in § 42. Thus in Figs. 34 and 37 the moments are positive, and in Figs. 35 and 36 negative.

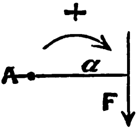


Fig. 34.

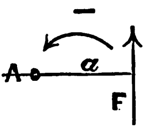


Fig. 35.

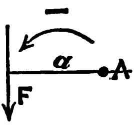


Fig. 36

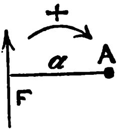


Fig. 37.

52. The following are the chief properties of moments :—

(i.) *The sum of the moments of a number of forces about any point taken with their proper signs is equal to the moment of the resultant about the same point.*

(ii.) *When any number of forces are in equilibrium, the sum of the moments with their proper signs about any point is zero.*

(iii.) *When two forces act on a particle the moments about a point in the direction of the resultant are equal.*

Example. A horizontal beam A B is attached, by a joint at its middle point C, to a vertical beam C D. A weight of 6 cwt. is suspended from A, and a force P, applied at B in a direction making an angle of 45° with the horizon, keeps A B horizontal. Find P.

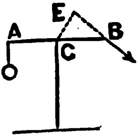


Fig. 38.

There are three forces acting on the beam A B (neglecting its weight). The weight at A, the force at B, and the reaction of the hinge at C. Let A C $= a$, then C B $= a$, and if we draw C E perpendicular to the direction of P, angle C B E $= 45°$, and C E $= \frac{a}{\sqrt{2}}$

Hence the moment of P about the point C $= \frac{P \cdot a}{\sqrt{2}}$

The moment of the weight about C $= -6\,a$.

The moment of the reaction about C $= 0$ (§ 50).

Therefore the sum of the moments is

$$\frac{P \cdot a}{\sqrt{2}} - 6\,a + 0 = 0$$

$$\therefore \quad P = 6 \cdot \sqrt{2} = 6 \times 1 \cdot 4142 = 8 \cdot 4852 \text{ cwt.}$$

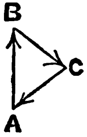


Fig. 39.

EXERCISES.

1. Show that when three forces act along the sides of a triangle, as in the figure, they cannot be in equilibrium.

2. Explain what is meant by the moment of a force about a point. State the principal properties of moments.

3. Show by the Principle of Moments that if three forces act on a body, and the directions of two of them pass through a fixed point, there cannot be equilibrium unless the direction of the third also passes through the fixed point.

4. A metal rod A B is bent at its middle point C, so as to form an angle of 105°. It is then laid across a smooth peg, and weights P and Q are attached to the ends. When the rod has assumed the position of rest, it is found that a plumb-line at C divides the angle A C B into angles of 60° and 45°. Prove that $P^2 : Q^2 :: 2 : 3$.

VII.—CENTRE OF GRAVITY.

53. The attraction of the earth which causes a body to have weight, acts on every particle of the body; thus, if we take a stone and pound it into small fragments, the sum of the weights of the small particles will be equal to that of the whole stone.

If one of these particles be attached by a fine thread to a fixed point o, the thread will take the direction of the vertical through o. If several of them be suspended from points near together, the threads will be parallel. When therefore the particles are united so as to form the body, we may regard their weights as a system of parallel forces.

Suspend the body by a point A (Fig. 40), the *resultant of the weights* of the particles will be equal to

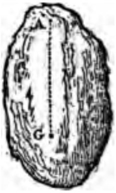


Fig. 40.

their sum, and will have the direction of the vertical through A. Suspend the body again from another point; the weight of each particle will have the same magnitude and the same point of application as before, but the direction of the forces with regard to the body will be changed. The result is the same as if each force had been turned about its point of application; hence the new line of support or direction of the resultant will intersect the old one in the centre of the forces. If the body were composed of a plastic material, and pierced in the direction of the line of support in several different positions, all the lines of perforation would intersect in a common point. This point is termed the centre of gravity of the body. We are led therefore to the following definition:—

54. *The resultant of a system of parallel forces acting on a rigid body passes through a fixed point the position of which is independent of the direction of the forces. If the forces be the weights of the several elements of the body, the fixed point is termed the centre of gravity.*

When a body is suspended from a point A it comports itself as if its weight were concentrated at the centre of gravity G. We may consider therefore that two forces act upon it—the resultant weight along the vertical through G and the reaction of the point of support along the vertical through A. When A and G are in the same vertical line the body is at rest; hence, if the C. G. be supported, the whole body will be supported.

We will now consider the position of the centre of gravity in certain figures.

The process of finding the centre of gravity of a number of isolated particles is precisely the same as that of finding the centre of a system of parallel forces.

We shall be concerned only with bodies through the volumes of which matter is distributed uniformly. Such bodies are termed homogeneous. A solid body is homogeneous when any two parts of equal volume are exactly of the same weight. The determination of the centre of gravity of a homogeneous body is therefore a purely geometrical question. The weights of different portions will be proportional to their volumes; hence we may treat the volumes as the forces.

Again, consider a very thin sheet of metal, or paper, of uniform thickness. The weights of any two portions will be proportional to the areas; hence we may treat the areas as forces, and seek the centre of gravity of the *surface.*

In like manner if we take a very thin wire of uniform thickness, the weights of different portions will be proportional to their lengths. We may therefore find the centre of gravity of a heavy line.

55. The following considerations will assist us in solving problems connected with the centre of gravity.

1. If a body be symmetrical about a plane, the C. G. lies in that plane. Every particle on one side corresponds to an equal particle on the other. Hence the C. G. of every pair of particles lies in the plane, and, therefore, so also does the C. G. of the whole.

2. It follows that if a body have two planes of symmetry, the *C. G. lies in* this line of intersection; and if it have three planes *of symmetry intersecting* in two lines, the C. G. is at the point *where the lines* cut one another.

3. If an area be symmetrical about a line, the C. G. lies in that line.

4. If a body have a centre of figure, that is, a point such that all lines drawn through it to the outline of the figure are bisected in the point, the centre of figure is the C. G.

For any two lines drawn through this point will contain figures on opposite sides of the vertex, in every respect equal. If any point in one of these figures be joined with the corresponding point of the other, the line drawn will be bisected by the centre, therefore the line joining their C. G.s will be bisected by the centre of figure. Consequently, this point is the C. G. of the two figures; similarly it is the C. G. of every other pair included by lines through the centre, and therefore it is the C. G. of the whole.

From these facts we may conclude at once that—

1. The C. G. of a straight line is its middle point.
2. The C. G. of the circumference or area of a circle is the centre.
3. The C. G. of the perimeter or area of a parallelogram is the point of intersection of the two diagonals, for this point is the centre of figure.
4. The C. G. of the volume or surface of a sphere is the centre.
5. The C. G. of a right circular cylinder is the middle point of the axis.
6. The C. G. of a parallelopiped is the point of intersection of two diagonals.

To find the C. G. of the Surface of a Triangle.

56. Let ABC be the triangle (Fig. 41). The C. G. may be found experimentally thus:— Suspend it by one of the angles, A, from a point O, and mark on the triangle the line AD *in the direction* of the

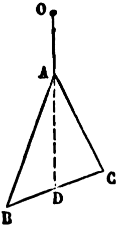


Fig. 41.

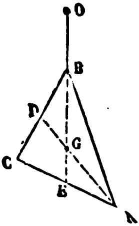


Fig. 42.

plumb-line through O. Take off the triangle and suspend it by another point B. Draw B E on the direction of the plumb-line, intersecting A D in a point G. The point is evidently the C. G. If the triangle be again suspended, the line of support will pass through G. It will be found that E and D bisect the sides A C and D C. Hence to find the C. G. of a triangle, join the middle points of two sides with the opposite angles.

57. We may, however, find the C. G. of the triangle by drawing only one middle line; for the point of intersection of the three middle lines of a triangle cuts off a third of each middle line.

Hence the C. G. of a triangle is in the line joining the apex with the middle of the base, and at one-third of its length from the base.

To find the C. G. of three equal Particles placed at the angles A B C of a Triangle.

58. The resultant of P at B and P at C will be 2P at D (Fig. 41), and the C. G. of 2 P at D, and P at A is at a point G, such that D G equals $\frac{1}{2}$ A G. Hence the C. G. of the three equal particles coincides with the C. G. of the triangle.

59. By this proposition the C. G. of a surface can be readily found when the surface can be divided into three equal figures. For instance, let it be required to find the C. G. of the portion of a square which is left after cutting off the two triangles formed by joining the middle point of one side with the middle points of two adjacent sides. The surface consists of a triangle *and two small squares*, each of which is one-fourth of *the large square.* Hence the C. G. of the whole is

found by joining the C. G.s of the three parts, and finding the C. G. of the triangle thus formed.

60. The centre of gravity need not be a point actually within the body, for instance, the C. G.s of the crescent and ring in Figures 43 and 44 are at G, and that of the two separate bodies in Figure 45 at C. The centre of gravity in these cases is that point which would possess the characters defined if the point were connected with the body by a straight rigid rod without weight.

61. The C. G. of two separate and independent bodies may be easily determined as follows :—Let the centres of gravity of the two bodies in question be connected by a straight line, and let a point be found upon this straight line which shall divide it into two parts, inversely proportional to the weights of the two bodies. Then this point will be their common centre of gravity.

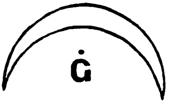


Fig. 43.

Thus, let M and N (Fig. 45) be the two bodies, and let *a b* be their centres of gravity. Draw the line *a b*, and take upon it a point C, so that *b* C shall bear to *a* C the same proportion as the weight of A bears to the weight of B. In this case, C will be the centre of gravity of the two bodies. Now if the line *a b* were a rigid rod, without weight, the point C would have all the properties of the centre of gravity; thus, the bodies would balance themselves on C in any position.

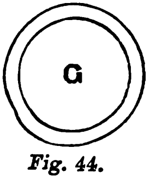


Fig. 44.

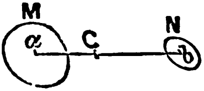


Fig. 45.

To find the C. G. of equal weights placed at equal distances on the circumference of a circle.

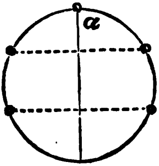


Fig. 46.

62. Apply § 55, or proceed thus:—From any one *a* of the weights draw a diameter, and join each weight on the right of this diameter with the corresponding weight on the left. The C. G. of each pair of weights thus joined will lie in the diameter from *a*, hence the C. G. of the whole will lie in this diameter. Similarly it may be shown that the C. G. lies in the diameter drawn from another weight *b*; hence it is the intersection of two diameters, and is therefore the centre of the circle.

63. As an illustration, let us consider the position of the C. G. of a round table standing on a level floor, and supported on three legs placed on the circumference (Fig. 47). By the preceding proposition the C. G. of the table coincides with the C. G. of three equal pressures at the angles of an inscribed equilateral triangle: consequently, the table will be supported if each leg bears one-third of the weight of the table. If

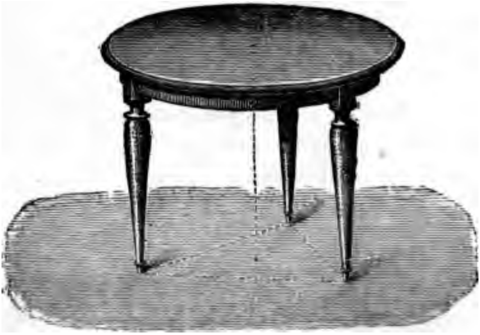


Fig. 47.

a weight be placed on the table in a position which is not the centre, it may be shown by § 41 that the pressure on each leg is proportional to the distance of the weight from the opposite side of the triangle formed by the legs. Hence, to obtain the pressures, we must find the point in which the vertical through the C. G. of the weight meets the floor (Fig. 47), and then divide the weight into parts proportional to the perpendiculars from this point to the sides of the triangle formed by the feet. Thus, for instance, if these distances be respectively 2, 3, and 4 feet, the pressures are respectively $\frac{2}{9}$ $\frac{3}{9}$ $\frac{4}{9}$ of the weight.

Properties of the Centre of Gravity.

64. We have seen that when a body is suspended from a point the vertical through that point must contain the C. G. of the body. The same is true when the body rests on a point. (Fig. 48.)

If the body rest on more points than one in the same horizontal plane, the resistances at these points will be parallel forces, and will therefore have a single resultant parallel to them. The direction of this resultant will lie within the base formed by the points; hence *the vertical through the C. G. of the body must fall within the base.*

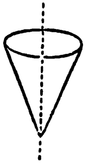


Fig. 48.

The force required to move a body may vary with the position of the body. Let A B C D (Fig. 49) be the section of a prism resting on a horizontal plane. Turn it about one edge A. The C. G. describes a circle, and the force required to move it decreases as the C. G. ascends; in other *words, the stability* of the body

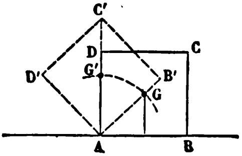


Fig. 49.

decreases as the C. G. is raised. When the C. G. arrives at the position G′ in the vertical through A, the body reaches the limit of stability. In this position the equilibrium is mathematically possible, for the resistance of the surface through A will be in the same straight line as the C. G. The slightest force will be sufficient to move the body, and when disturbed it will fall away from this position.

65. When a body in equilibrium would return to its original position if slightly displaced, the equilibrium is said to be *stable*; when the body would fall away from its first position if slightly displaced, the equilibrium is said to be *unstable.*

Remark that there is equilibrium only when the C. G. occupies the lowest or highest possible position, and the equilibrium is stable in the first position and unstable in the second. Hence every body suspended from a point is in stable equilibrium, and every body supported above a point or line is in unstable equilibrium.

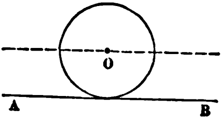


Fig. 50.

Consider now a sphere resting on a horizontal plane, the reaction of the plane and the weight of the body will act along the same straight line in every position. If therefore the sphere be slightly displaced it will have no tendency either to return to or move away from its first position. Its C. G. will neither be raised nor lowered. The equilibrium in this case is *neutral.* *When the body* rests on

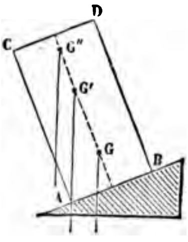


Fig. 51.

an inclined plane (Fig. 51), it is still necessary that the vertical through its C. G. shall fall within the base. Let C D represent a prism, and A B its base on an inclined plane; the resultant of the resistance offered by the points in contact perpendicular to the plane will evidently pass through a point in A B. If the body be prevented from sliding by the roughness of the plane, the perpendicular reaction of the plane and the force caused by this roughness meet in a point in A B, and have therefore a resultant through the same point. That this resultant may be in equilibrium with the weight, the vertical through the C. G. of the prism must evidently fall within A B. If the C. G. be at G, the equilibrium will be stable, if at G′ unstable, and if at G″ the body will fall.

66. The necessity of keeping the C. G. so that the vertical through it shall fall within the base formed by the points of support has constantly to be considered. When a man stands or walks, the line of direction of his weight must fall within the base formed by his feet. If he take a load in one hand, the C. G. of the whole figure will be thrown towards the load, and he naturally raises the free arm and leans towards the other side, thus making the new C. G. fall over the base formed by his feet. If he carry the burden on his back, he bends forward; if he bear it in front, he leans back. In every case the load and the bearer form a whole and the C. G. of the whole is displaced on

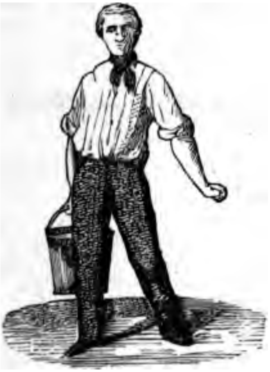


Fig. 52.

the side of the load so that it is necessary to bring the new C. G. over the base of support.

Consider now a cart with two wheels; if the vertical through the C. G. of the loaded cart fall in the line joining the points of contact of the wheels and ground, the weight is supported by the road; in descending an incline the vertical through the C. G. will fall in front of this line, and part of the weight must be borne by the horse; in an ascent the C. G. falls behind, and the load tends to lift the horse. When the cart is properly loaded on a horizontal road, the vertical through the C. G. falls near the middle of the line between the wheels, but on a road inclined towards one side the vertical falls nearer the lower wheel. So long as it does not pass the lowest point of this wheel equilibrium is maintained, but as soon as it passes it the cart is overturned The higher the C. G. is above the ground the sooner is this position reached.

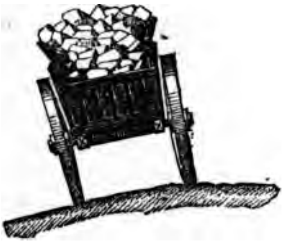


Fig. 53.

To find, geometrically, the C. G. of any rectilineal figure.

67. Consider first a quadrilateral, A B C D (Fig. 54). Join A C, and find G_1, G_2, the C. G.s of the triangles A B C, A D C. The C. G. of the whole must be in the line G_1 G_2. Now join B D, find G_3 G_4 of the triangles B A D, B C D: the C. G. of the whole must be in the line G_3 G_4. It is therefore *at their* point of intersection.

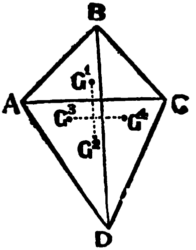


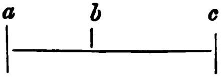
Fig. 54.

If the figure have five sides we can divide it in two ways, into a triangle and a quadrilateral, and join their C. G.s.

From a pentagon we may proceed to a hexagon, and so on.

68. *Example 1.*—Find the centre of gravity of three spheres, A of 3 lbs., B of 5 lbs., and C of 7 lbs., the centres of A and B being 6 inches apart, and those of B and C 9 inches apart.

Let a, b, and c be the positions of the three centres. The weights of the spheres may be regarded as parallel forces acting at a, b, and c. Taking a as the fixed point from which distances are measured, and forming the products according to the rule in § 41, we have, if x stand for the distance of the C. G. from a,



$$3 \times 0 + 5 \times 6 + 7 \times 15 = 15 \cdot x.$$

$$\text{or} \quad 135 = 15\,x$$

$$\therefore \quad x = 9.$$

69. *Example 2.*—A beam weighing 200 lbs. is carried by two men, one, A, being at a distance of 3 feet, from the centre of gravity of the beam, and the other, B, at a distance of 2 feet. Find the weight borne by each man.

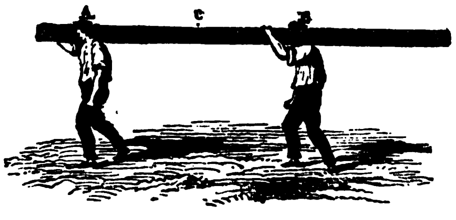


Fig. 55.

Let P be the pressure sustained by A, then 200 — P is the pressure sustained by B, and if O be the C. G.

$$3P = 2\ (200 - P)$$
$$5P = 400$$
$$P = 80\text{ lbs.}$$

The pressure sustained by $B = 200 - P = 120$ lbs.

EXERCISES.

1. Find the C. G. of two spheres weighing respectively 8 lbs. and 3 lbs., the distance between their centres being 2 ft. 9 in.

2. Find by a geometrical construction the C. G. of a bent wire, one arm being twice the length of the other.

3. Find the C. G. of two solid bodies of 12 lbs. and 20 lbs. respectively, the line joining their centres of gravity being 2 ft. 8 in.

4. The distance of the C. G. of two heavy particles from the greater is 5 inches, the particles are respectively 60 and 72 grammes; find the distance between them.—*Ans.* 11 in.

5. Weights of 2 lbs., 4 lbs., 6 lbs., and 8 lbs. are placed so that their C. G.s are in a straight line, and 6 inches apart; find the distance of their common centre of gravity from that of the largest weight.—*Ans.* 6 in.

6. Find the C. G. of the remainder of a square when one of the triangles into which the diagonals divide it is taken away. (§ 58)—*Ans.* If a = the side of the square the C. G. is $\frac{a}{9}$ from the centre.

7. A B C D is a quadrilateral in which the base A D is double each of the other sides; find the C. G. (§ 58)—*Ans.* $\frac{4}{9}$ths of the height from the base.

8. If three men support a triangular board at its three corners, what portion of the weight will they bear?

9. If from one of the angles of a triangle a weight be suspended equal to the weight of the triangle, where will be the centre of *gravity?*

10. If at two of the angles of a triangle weights be placed each equal to half the weight of the triangle, where will be the common centre of gravity?

11. Weights of 2 lbs., 3 lbs., 4 lbs. and 5 lbs. respectively are placed at the corners of a square whose side is 4 ft. Find the distance of the common C. G. from the centre.—*Ans.* $\frac{4}{7}$ths of a foot.

(Find the distance of the C. G. of 2 and 5 from the middle of the side 2—5, and also the distance of the C. G. of 3 and 4 from the middle of the side 3—4, and show that the distance of the common C. G. from the centre is the arithmetical mean of these distances.)

12. On a circular table with three legs, placed at equal distances on the circumference, a weight of 94 lbs. is placed, so that the distances of the C. G. of the weight from the sides of the triangle formed by the legs are respectively 4, 3, and 2·4 feet. Find the pressure of the weight on each leg.—*Ans.* 40, 30, and 24 lbs.

13. A weightless plate in the form of an equilateral triangle is suspended by three parallel strings attached to its angular points. The strings can support weights of 1 lb., 2 lbs., 3 lbs., respectively. Where on the plate must a weight of 6 lbs. be placed so as to be supported?

Ans. The position may be found by the following construction. Join the middle points of two sides, and divide the line drawn into parts having the ratio 1 : 2.*

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* If any point be taken in an equilateral triangle, the sum of its distances from the sides is equal to the height of the triangle.

VIII.—THE MECHANICAL POWERS.

Machines.

70. An instrument for transmitting a force from one point to another, or for changing the direction of motion, is a machine.

Any machine, however complex, is composed of a number of parts termed the simple mechanical powers.

Let us consider, for example, a flour mill. (Fig. 56.) On the left of the figure is the water-wheel, to which motion is first communicated. The moving force of the water is transmitted by the axle A to the first vertical toothed wheel, and by it to the horizontal toothed wheel attached to the axle B, which also carries a large toothed wheel C. This wheel communicates with a smaller one on the same shaft as the millstone, capable of being moved up and down upon the shaft. Thus there are three distinct parts to the machine; (1) the axle A, with its two wheels, the water-wheel to which the moving power is applied, and the toothed wheel to which is applied the resistance to be overcome; (2) the axle B, to the lower wheel of which the power is applied, and to the upper the resistance; (3) the axle N with a power-wheel moved by the wheel C, and the millstone to which is applied the final resistance. Now from the wheel which receives the force of the water to the stone which grinds the corn, the parts of the machine are so connected that the uniform movement of one part secures a like movement of the whole, and the equilibrium of the whole implies the equilibrium of each part. To *trace the conditions* of equilibrium of such a machine

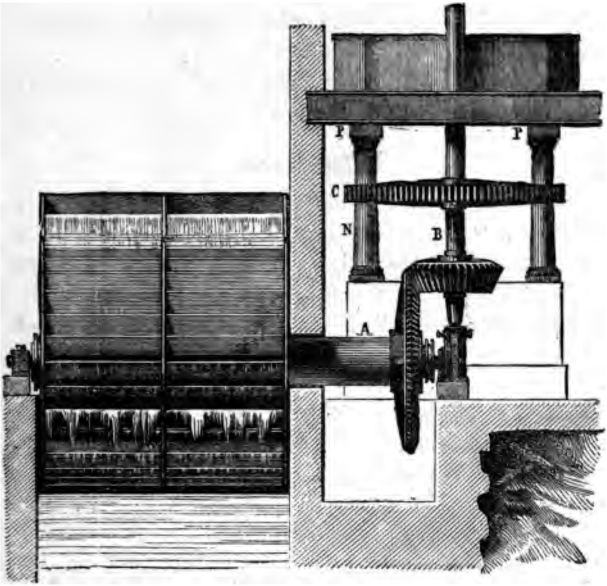


Fig. 56.

it is only necessary therefore to study the equilibrium of the simple machines.

71. It will be expedient to omit in this investigation several things which must be considered before a practical application of the principles can be made. For example, we shall first take no account of the friction between the different parts of the machine. Except where the contrary is stated, we shall neglect the weight of the machine itself. Cords and chains will be treated as if they had neither stiffness, thickness, nor weight. The consideration of these resistances is for the present *deferred*.

72. The force applied is technically called the *power*, and the resistance, whatever may be its nature, is termed the *load*, or *weight*.

The condition of equilibrium in each machine will be expressed by stating the ratio of the power to the weight.

The Lever.

73. Any rigid rod movable about a fixed point is termed a *lever*. That there may be equilibrium with such a machine the forces acting on it must admit of a resultant passing through a fixed point in the lever. This point is termed the *fulcrum*, and the resultant of the force is the pressure sustained by the fulcrum.

Levers are usually classified according to the position of the fixed point. If the fulcrum be between the power and the weight, the lever is of the *first* kind. If the weight be between the fulcrum and the power, the lever is of the *second* kind. If the power be between the fulcrum and the weight, the lever is of the *third* kind.

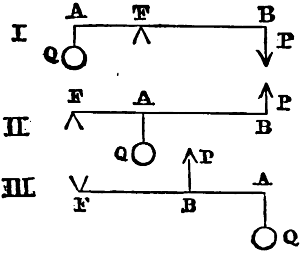


Fig. 57.

The crowbar of the mason, used as in Fig. 58, is a lever of the first kind.

The loaded barrow raised, as in Fig. 59, is a lever of the second kind.

The treadle of a sewing-machine is a lever of the third kind.

First, let us suppose the power and weight to be parallel forces.

Let P and Q be respectively the power and the weight applied at the points A and B. Let O be the fulcrum; and let A O $= a$, B O $= b$. Then, by the principle of

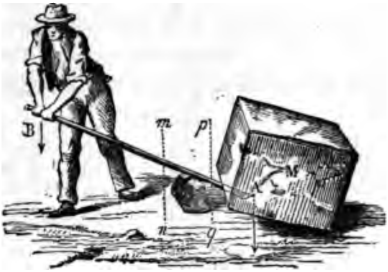


Fig. 58.

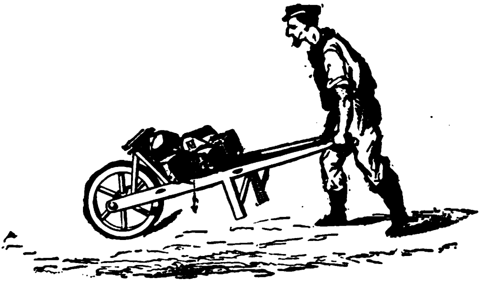


Fig. 59.

parallel forces, whichever kind of lever we consider, the equation of equilibrium will be

$$P \cdot a = Q \cdot b.$$

The pressure on the fulcrum in the lever of the first kind (Fig. 57 I.) is P + Q; in the lever of the second kind (Fig. 57 II.), Q — P; in that of the third kind (Fig. 57 III.), P — Q.

In all cases the greater force is attached to the shorter arm, so that Q is greater or less than P, according as *a* is greater or less than *b*. Hence, in lever 1, P may be greater or less than Q; in lever 2, since *a* is the whole rod P is necessarily less than Q; and in lever 3, P is necessarily greater than Q. If the forces be not parallel (Fig. 60) still the moments about the fulcrum in opposite directions must be equal, hence $P \cdot p = Q \cdot q$. To find the magnitude and direction of the pressure on the fulcrum O, suppose the forces P′ and Q′ at O equal and parallel to P and Q, then R, the resultant of these forces, will be the pressure on the point O.

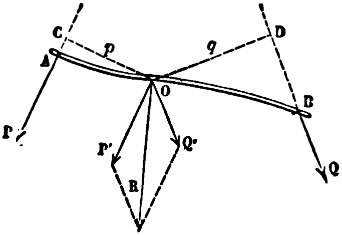


Fig. 60.

Balances.

74. The balance is a lever of the first kind, having equal arms. The ordinary balance consists of a metallic bar termed the *beam* (Fig. 61), supported on a stand (*c*) by a sharp edge, called a knife edge, fixed horizontally in the middle of the beam. Attached to the *beam* at right angles is a needle or index, which moves

over a graduated arc, and is vertical when the beam is horizontal. Two dishes or scales (*b*, *b*) for receiving the weights are suspended from the extremities of the beam, also upon knife edges. These edges form the points of application of the weights. The distances of these points from the central edge are the arms of the balance. When equal weights placed in the scales always cause the beam to remain horizontal the balance is *true*. When this is not the case the balance is *false*.

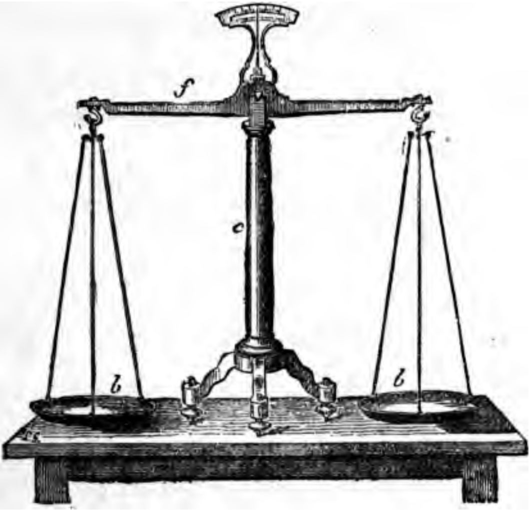


Fig. 61.

To investigate the requisites of a good balance.

75. 1. When the scales are empty, the beam should be horizontal and index vertical. If the balance be

disturbed, the index should return again to the same place when the motion of the beam has subsided; hence the equilibrium should be stable. To secure this *the C. G. of the beam and its appendages should fall a little below the knife edge.*

2. When equal weights are placed in the scales the index should be vertical; hence *the arms must be of the same length.*

3. When the weights are unequal, the balance should easily indicate the inequality; in other words, the balance should be sensitive.

76. The sensibility of the balance depends on two things,—it is increased by augmenting the length of the arms and is diminished by increasing the weight of the beam.

77. To test a balance, all that is necessary is to take weights apparently equal, and to transfer them. If they be not really equal, but appear so only in consequence of the inaccuracy of the balance, the fact will be made apparent when each is moved to the opposite scale.

The exact weight of a body may, however, be found by the aid of a false balance. Place the body to be weighed in one scale, and as much of some substance (sand for example) in the other as will make the beam horizontal. Now take out the body, replacing it by such weights of known value as will balance the sand. The sum of these weights will be the weight of the body. We may, however, find the true weight of a body by weighing it in both scales. Let a and b be the arms of the balance, w the true weight of the body, x the weight at b which balances w at a, and y the weight

at *a* which balances W at *b*. Then $W \cdot a = x\,b$
and $W \cdot b = y \cdot a$

On multiplying these equations we find that

$$W = \sqrt{xy}.$$

78. The common steelyard is a balance with unequal arms (Fig. 62). On one side of the knife-edge O, which is the fulcrum of the lever, the weight P remains the same, but its distance OM varies; on the other side, the weight suspended from the edge A may be changed, but the distance AO is constant. The condition of equilibrium, if we suppose the C. G. of the machine to be at O, will be $Q \cdot AO = P \cdot OM$. As Q varies on the one side, OM may be made to vary on the other, so that the same weight P may balance various weights suspended from A. For example, let P in one position M balance 1 lb.; then, in a position M′ such that $OM' = 2 \cdot OM$, P will balance 2 lbs. If the C. G. of the bar

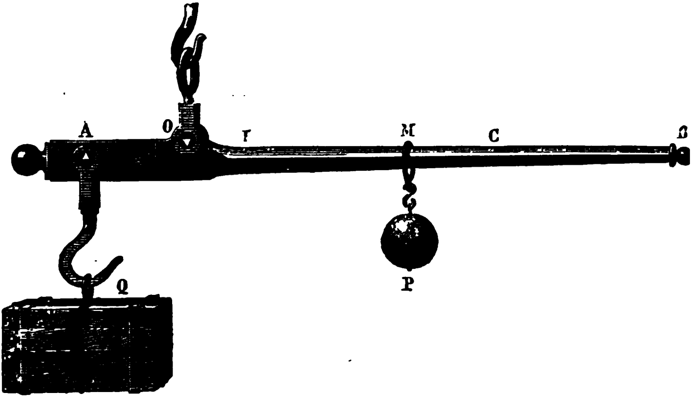


Fig. 62 and 63.

and its appendages be not at O, but at G, for example, then the moment of the weight of the bar W will be in the same direction as that of Q, and the condition of equilibrium will be $Q \cdot AO + W \cdot GO = P \cdot MO$.

79. In the Danish balance (Figs. 64 and 65), the power is the weight P of the lever acting at its centre of gravity, and the ratio of the arms is altered by moving the fulcrum when the lever is straight (Fig. 64), or altering the inclination when the lever is bent. In this case there is equilibrium when $P \times af = W \times bf$.

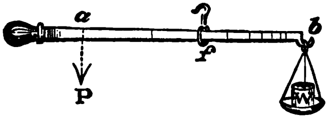


Fig. 64.

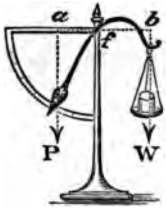


Fig. 65.

80. The wheel and axle is only a modification of the lever. This machine consists of two cylinders, A B and C D, of different radius (Fig. 66), having a common

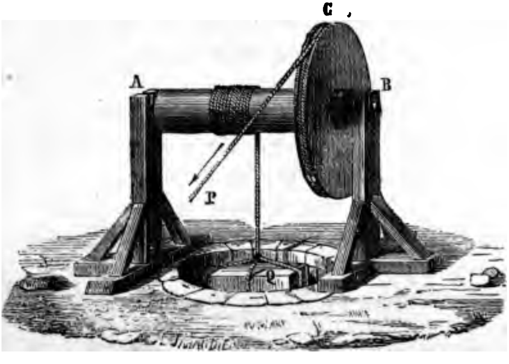


Fig. 66.

axis; the smaller being termed the axle, and the larger the wheel. A cord is wound round the wheel in one direction, and another cord round the axle in the opposite direction. To the former is attached the weight Q, and to the latter the power P. Suppose P and Q to be both vertical and represent the instrument as seen in the direction of the axis (Fig. 67). We have two parallel forces, P and Q, acting at the extremities of two arms, OB, OA, of a lever. The condition of equilibrium is therefore $P \cdot OB = Q \cdot OA$, or, if R and r be radii of wheel and axle, $P \cdot R = Q \cdot r$.

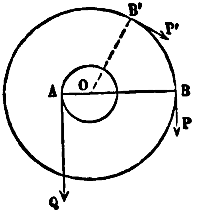


Fig. 67.

If the power does not act vertically but in the direction of the tangent at another point, as P' (Fig. 67), then $P' \cdot R$ is the moment of P' about O, and since $P'R = PR$, in this case also

$$P' \cdot R = Q \cdot r.$$

81. Instead of a wheel, one or several levers, l, l, (Fig. 68), may be placed in the axle as in the *windlass*.

82. When the axle is vertical, and the levers horizontal, the machine is termed a *capstan* (Fig. 69).

Toothed Wheels.

83. Several wheels and axles are frequently combined by means of spur wheels (Fig. 70). Several conditions must be satisfied in each pair of wheels. The teeth on each wheel must be equal to one another,

and equally distant, and the teeth of one wheel should be of the same size, and as far apart as those of the

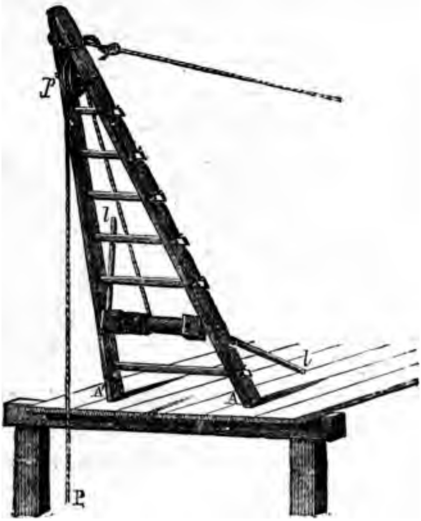


Fig. 68.

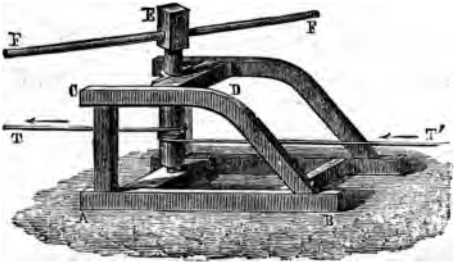


Fig. 69.

wheel in contact with it. The number of teeth will therefore be proportional to the circumferences or the radii of the wheels.

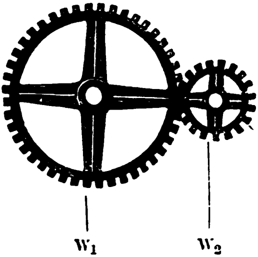


Fig. 70.

Equation of equilibrium for two-toothed wheels.

Suppose two such wheels to be in equilibrium with a weight W at the axle of the larger, and W_2 at the axle of the smaller. Let the radii of the wheels be R_1 and R_2, and let the radii of both axles be r. Also let P be the reaction between the wheels at their point of contact.

The equilibrium of the larger wheels gives $W_1 \cdot r = P \cdot R_1$

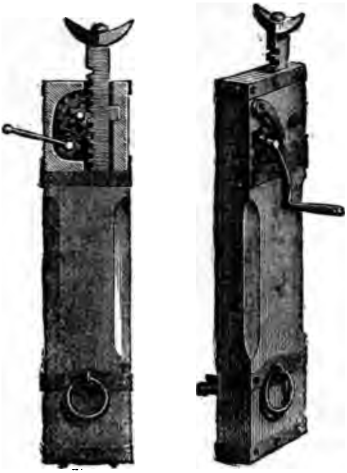
The equilibrium of the smaller . . . $W_2 \cdot r = P \cdot R_2$

therefore $\dfrac{W_1}{W_2} = \dfrac{R_1}{R_2} = \dfrac{\text{No. of teeth in larger.}}{\text{No. of teeth in smaller.}}$

84. Sometimes the teeth are placed on a straight bar instead of a wheel. Such a toothed bar is termed a *rack*, and the wheel in contact with it is called a *pinion.*

The *jack* (Fig. 71), used for lifting great weights through a small height, is a rack and pinion. The handle by which the rack is raised is termed a winch.

The instrument is an example of a double wheel and



Figs. 71 and 72.

axle; the winch takes the place of the first wheel, the lower pinion being its axle. The upper pinion is the second axle and supports the rack. The large toothed wheel of the upper axle works in gear with the lower pinion. The two pinions are usually, as in the figure, of the same size.

To find the equation of equilibrium.

Let P be the power applied to the winch, W the weight of the rack and its load, a the radius of the winch, b the radius of the large toothed wheel, r the radius of each pinion, and R the mutual reaction between the first pinion and the large wheel; then from the equilibrium of the first axle we have $P \cdot a = R \cdot r$, and from the equilibrium of the second axle we have $W \cdot b = R \cdot r$. *Therefore* $P \cdot a = W \cdot b$.

EXERCISES.

1. A weight of 30 lbs. balances a weight of 20 lbs. at the extremities of a straight lever 15 feet long; find the length of the arms.—*Ans.* 6 feet, 9 feet.

2. The arms of a lever are 7 in. and 9 in. in length, and the weight 3 lbs. is attached to the shorter arm; find the power.—*Ans.* 2⅓ lbs.

3. If one end of a bar rest on a beam and a weight of 50 lbs. be suspended from it at one-fifth of its length from the beam, what power at the other end will support the weight, and what will be the pressure on the beam?—*Ans.* 10 lbs., 40 lbs.

4. A beam 4½ ft. long is supported horizontally by two props at its extremities and produces a pressure of 4 lbs. on each prop; where must a weight of 36 lbs. be placed that the whole pressure on one prop shall be 10 lbs.?—*Ans.* 9 inches from one prop.

5. When two weights, 15 lbs. and 5 lbs., are suspended at the ends of a lever, the fulcrum is 9 feet from the smaller weight; where must the fulcrum be when the weights are each increased 5 lbs.?—*Ans.* 4 ft. from one end.

6. The C. G. of a wheelbarrow and its load, which weigh 100 lbs., is in a vertical line 18 inches from the centre of the wheel; what power applied to the handles at a distance of 3 ft. 6 in. from the centre of gravity will just lift the barrow?—*Ans.* 30 lbs.

7. A beam 12 ft. long balances about its middle point; about what point will it balance if a weight equal to twice that of the beam be placed at one end?

8. A beam A B 10 ft. long and weighing 56 lbs. balances about a point 3 ft. from A. When a weight is placed at B, the beam balances about a point 1·4 feet from B; find the weight.—*Ans.* 224 lbs.

9. The pressure on the fulcrum is 7, and the sum of the forces 13; find their distance from the fulcrum when the forces are 14 inches apart.—*Ans.* 6 in. and 20 in.

10. Find the true weight of a substance which, when placed in

one scale of a balance, seems to weigh 140 grammes, and in the other appears to weigh 154·35 grammes.—*Ans.* 147 grammes.

11. If in a balance one arm be ·98 of the other, and a body placed in the scale of the shorter arm balance 14·7 kilogrammes in the other scale, find the true weight of the body.—*Ans.* 15 kilogrammes.

12. The beam of a false balance is attached to one arm of a true balance, and weighs 2 lbs. A body placed in one scale of the false balance requires a weight of 3 lbs. in the other, and then the whole weighs 7·9 lbs.; find the weight of the body and the ratio of the arms.—*Ans.* 2·9 lbs.; 30 : 29.

13. The weight of a steelyard is 1 lb., the movable weight also 1 lb., the point of suspension of the body 8 inches, and the C. G. of the beam 3 in. from the fulcrum; graduate the beam for weights from 1 to 12 lbs.

14. What effect is made on the graduations by increasing the movable weight?

15. With a wheel and axle a power of 8 lbs. sustains a weight of 12 lbs.; what is the radius of the axle, that of the wheel being 24 in.?—*Ans.* 16 in.

16. The circumferences of wheel and axle are respectively 1 yard and 15 inches; what power will sustain a weight of 1½ tons?—*Ans.* 12·5 cwts.

17. Find the pressure on the catch of a ratchet-wheel (Fig. 72) 12 inches in diameter when it is attached to an axle 5 inches in diameter, sustaining a weight of 60 lbs.—*Ans.* 25 lbs.

18. A capstan turned by two horses is used to draw in a boat; the levers to which the horses are attached are 12 feet long, and the radius of the axle is 18 inches. When each horse is pulling with a force of 7½ cwts., find the tension of the cord attached to the boat.—*Ans.* 6 tons.

19. A uniform bent lever, the weights of whose arms are 5 lbs. and 10 lbs., rests with the shorter arm horizontal, what weight must be attached to the end of the short arm that the lever may rest with the long arm horizontal?—*Ans.* 37½ lbs.

The Pulley.

85. A pulley is a circular disc of metal or wood, capable of turning round an axis passing through its centre. Usually a groove is cut in the disc to keep a cord which passes over the pulley from slipping off (Fig. 73). Sometimes, however, the cord is replaced by a strap, and then the edge is convex (Fig. 74).

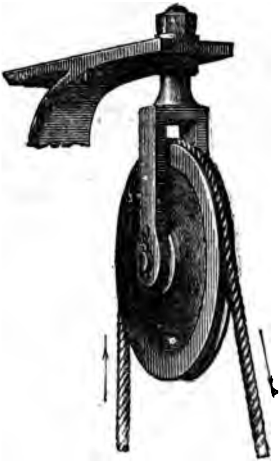


Fig. 73.

The pulley may be considered as a lever with equal arms, A O, O B (Fig. 75), so that forces P and P′ in equilibrium, at the extremities of the cord, are equal.

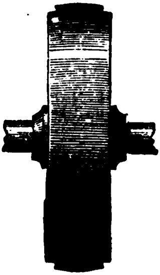


Fig. 74.

Whether the two parts of the cord be parallel or not these forces are equal, so that if one end of the cord be attached to a dynameter (Fig. 76), the indication of

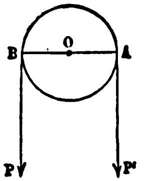


Fig. 75.

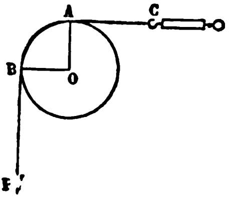


Fig. 76.

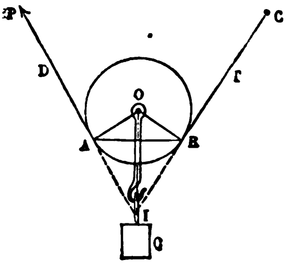


Fig. 77.

the instrument will be the same as if a weight equal to P were attached to it directly. A pulley, the axis of which is fixed in space (Fig. 73), is termed a *fixed* pulley, and in mechanics serves the purpose only of changing the direction of the power. If the axis be movable, the pulley is termed a *movable* pulley.

Let such a pulley with a weight Q attached, be supported by a cord C B A D (Fig. 77), fixed at a point C, and having at the other extremity a force P. The cord B C exerts a certain pressure T on the fixed point C, and this point reacts with a force equal and opposite. The pulley is then supported by the three forces P, T, and Q, which may be supposed to act at the points A B and O. That there may be equilibrium it is necessary that the directions of these forces shall meet in one point (§ 29). Let I be the point in the direction of Q, in which P and T meet.

The triangles O A I and O B I are equal in every respect, and therefore O I bisects the angle A I B. It follows that the forces P and T are equal, and, consequently, if we know the angle I, and the weight Q, we can find P, or if we know P and the angle I, we can find Q. When the cords A D, B C are parallel, Q is the resultant of two equal parallel forces, and therefore $2\,P = Q$. In this case the pressure on the fixed point C is also equal to $\frac{1}{2}$ Q.

Several movable Pulleys with separate strings.

86. In this system of pulleys each movable pulley has a string of its own, one end of which is attached to the beam, and the other end to the next pulley, the power being applied to the free end of the cord passing over the highest pulley (Fig. 78). If we neglect the weights of the pulleys, and take T, T′, T″ to represent the tension of each separate cord, then

$$T = \frac{1}{2} Q ;\ T' = \frac{1}{2} T = \frac{1}{4} Q ;\ T'' = \frac{1}{2} T' = \frac{1}{8} Q$$

$$\text{and } T'' = P ;\ \text{hence } P = \frac{1}{8} Q.$$

If there be n movable pulleys, then $P = \frac{1}{2^n} \cdot Q$.

If we are required to take into account the weights W W′ W″ of the pulleys, then

$$T = \frac{1}{2} (W + Q); T' = \frac{1}{2} (W' + T); P = \frac{1}{2} (W'' + T')$$

therefore if there be n pulleys

$$P = \frac{1}{2^n} Q + \frac{1}{2^n} W + \frac{1}{2^{n-1}} W' + \frac{1}{2^{n-2}} W'', \&c.$$

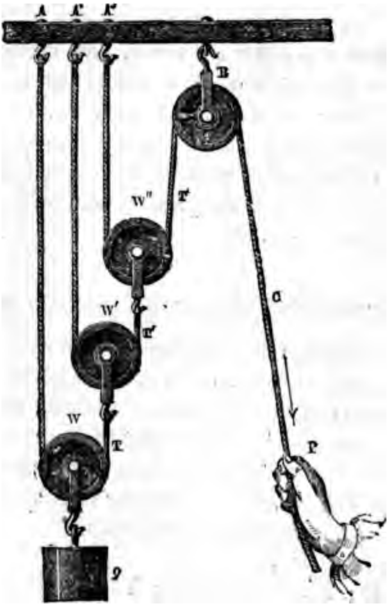


Fig. 78.

Several movable Pulleys in one block, the same string passing round all.

87. In this case we suppose the parts of the string between the pulleys to be parallel. The tension of the string is the same throughout, and is equal to P; hence, if, as in the figure (Fig. 79), there are three movable *pulleys*, *six* parts of the string support the weight,

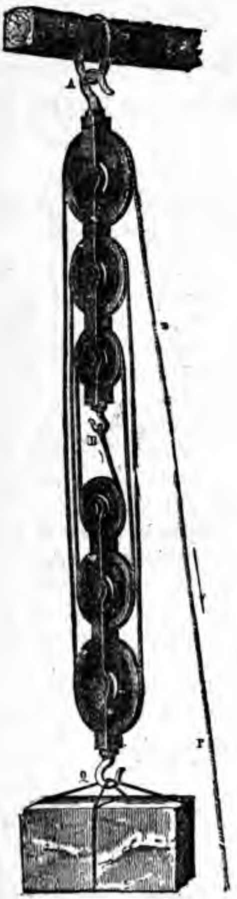


Fig. 79.

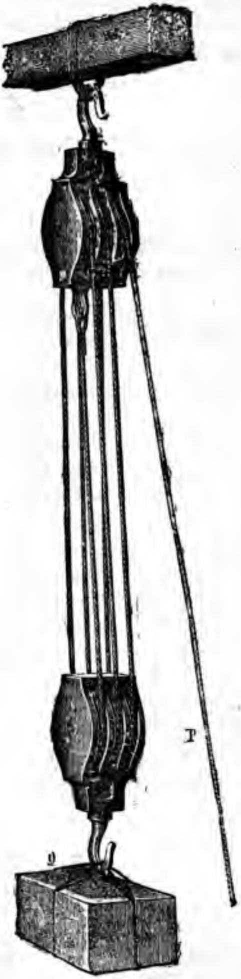


Fig. 80.

each bearing $\frac{1}{8}$; hence $P = \frac{1}{8} Q$. If there be n strings from the lower block

$$P = \frac{1}{n} Q.$$

The weight of the pulley will be taken into account by adding it to Q.

Several Movable Pulleys with Separate Strings, one end of each being attached to the Weight.

Let P be the power and W the weight. The tension in the string which passes over the first pulley A is P, that in the string over B supports the weight of the pulley A, and the tensions in the two parts of the string over A. If we neglect the weights of the pulleys, the tension in the string over B is 2 P. Similarly, the tension in the string over C is 4 P, and so on. Now all the strings support the weight, consequently W = the sum of the tensions $= P(1 + 2 + 2^2 + 2^3 + \&c.)$ By summing this geometrical progression, we obtain $W = P(2^n - 1)$, n being the number of strings attached to the weight.

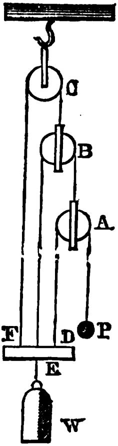


Fig. 81.

If the weights of the pulleys w_1, w_2, w_3, &c. are taken into account, we find the several tensions to be

$$\begin{aligned} T_1 &= P \\ T_2 &= 2P + w_1 \\ T_3 &= 2T_2 + w_2 \\ &= 4P + 2w_1 + w_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

$$T_n = 2^{n-1} P + 2^{n-2} w_1 + \dots\dots + w_{n-1}.$$

$$\text{Hence } W = \begin{cases} P \\ + 2P + w_1 \\ + 4P + 2w_1 + w_2 \\ + 8P + 4w_1 + 2w_2 + w_3, \text{ and so on.} \end{cases}$$

EXERCISES.

1. Show that with a single pulley a given weight is sustained by the least power when the parts of the string are parallel. (Fig. 77.)

2. Find the power which will support a weight of 611 lbs. with three movable pulleys, having separate parallel cords attached by one end to the beam, the pulleys taken in order from the lowest weighing respectively 7, 5, and 3 lbs.

$$\begin{array}{r} 611 \\ 7 \\ \hline 2)618 \\ \hline 309 \\ 5 \\ \hline 2)314 \\ \hline 157 \\ 3 \\ \hline 2)160 \\ \hline 80 \\ \hline\hline \end{array}$$

The formula of § 86 shows that the power will be found by adding the weight of the lowest block to the load Q, and dividing by 2; adding the weight of the next pulley, and again dividing by 2; and so on for all the blocks.

3. In the last example if the weights of the pulleys are 8, 6, and 4, and P = 34 lbs., find the load.

$$\begin{array}{rl} 34 & \times\ 2 \\ \hline 68 & \\ 4 & \\ \hline 64 & \times\ 2 \\ \hline 128 & \\ 6 & \\ \hline 122 & \times\ 2 \\ \hline 244 & \\ 8 & \\ \hline 236 & \\ \hline\hline \end{array}$$

Here the process is reversed, instead of dividing by 2 we multiply; instead of adding the weights of the pulleys we subtract them, commencing with the highest.

4. What power will sustain 1 ton with a block and tackle, the block containing 3 sheaves (pulleys) ?—*Ans.* $\frac{1}{6}$th ton.

5. What is the weight of a block containing 4 sheaves, by means *of which 20 lbs.* will balance 140 lbs.?—Ans. 20 lbs.

6. If 4 movable pulleys are used with the arrangement represented on Fig. 78, and the weights, taken in order from the lowest, are 7, 3, 8, and 5 lbs., find the power which will sustain a weight of 395 lbs.—*Ans.* 30 lbs.

7. In the above case, find the pressure on the beam.—*Ans.* 448 lbs.

8. Which would be the most advantageous arrangement of the 4 pulleys?

9. A man weighing 154 lbs. sustains a weight of 133 lbs. with 3 movable pulleys, each having a separate string attached at one end to the beam. The pulleys weigh respectively 3 lbs., 2 lbs., and 1 lb.; find the pressure exerted by the man on the floor on which he stands.—*Ans.* 111 lbs.

10. If the man in the above example exerts a pressure on the floor of 100 lbs., what weight does he sustain?—*Ans.* 421 lbs.

11. Find the power which will sustain a weight of 90 lbs. with a single movable pulley C, the cord making an angle of 60°.—*Ans.* $30\sqrt{3}$.

12. With a system of pulleys in which a separate string passes round each pulley, and is attached by one end to the weight, show that if we neglect the weights of the blocks the tensions in the different strings are

$$P, 2P, 4P, 8P, \&c.$$

$$\text{and that } W = P\,(1 + 2 + 2^2 + \&c.)$$

13. In the above system if there be n pulleys

$$W = P\,(2^n - 1).$$

14. If $W = 147$ lbs., $A = 1\frac{1}{2}$ lbs., and $B = 2\frac{1}{2}$ lbs., find P. —*Ans.* 20 lbs.

15. If in this system there be 5 pulleys, and the weights of the 4 movable blocks commencing with the lowest be 2, 3, 4, and 5 lbs., find the weight which will be sustained by a power of 12 lbs.—*Ans.* 440 lbs.

The Inclined Plane.

88. When a body is pressed against a hard smooth surface, the resistance offered by the surface is at right angles to it. Hence, a body may be supported on a smooth horizontal surface, for the weight acting downwards may be counteracted by the reaction of the surface acting upwards. A body cannot, however, rest on an inclined plane without the action of a third force. When the body is thus supported on a smooth inclined plane, the three forces must pass through the same point, and the resultant of any two must be equal and opposite to the third (§ 20). We may therefore apply the Parallelogram of Forces to find equations expressing the relations existing between the forces. These relations are very simple in the following particular cases:—

1. *When the force* F *acts in a direction parallel to the plane.*

Let B A C be the plane, and let *a* be the centre of gravity of the weight W. The direction of the weight is the vertical *a b* through *a*, and that of the reaction of the plane is *a c*, perpendicular to the plane. Now we may take any length along the vertical to represent the weight W. Let *a b*, the length taken, be equal to A B, to which the force is parallel. Through *b* draw *b c*, parallel to the direction of F; then *b c* represents F, and *a c* represents R, the reaction of the plane. Now the triangle *a b c* has its angles respectively equal to those of the triangle A B C, and a side *a b* equal to A B, therefore the triangles are equal in every respect; but by the triangle of forces, since there is equilibrium,

$$\frac{\mathrm{F}}{\mathrm{W}} = \frac{b\,c}{a\,b} = \frac{\mathrm{B\,C}}{\mathrm{A\,B}} = \frac{\textit{height}}{\textit{length}}$$

$$\frac{\mathrm{R}}{\mathrm{W}} = \frac{a\,c}{a\,b} = \frac{\mathrm{A\,C}}{\mathrm{A\,B}} = \frac{\textit{base}}{\textit{length}}$$

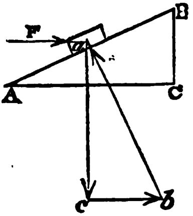


Fig. 82.

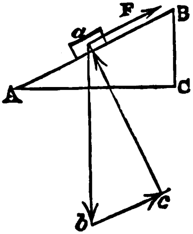


Fig. 83.

Example.—Find the power which, when acting parallel to the plane, will support a weight of 112 lbs., the height of the plane being 2 when the length is 7.

Here W = 112 lbs., height = 2, and length = 7.

Also $$\frac{\mathrm{F}}{\mathrm{W}} = \frac{\textit{height}}{\textit{length}}$$

$$\therefore \quad \frac{\mathrm{F}}{112} = \frac{2}{7}$$

$$\therefore \quad \mathrm{F} = \frac{2}{7} \times 112 = 32 \text{ lbs.}$$

2. *When the Force is horizontal.*

As before, let B A C be the plane, and let *a* be the C. G. of the weight W. The direction of the weight is the vertical *a c* through *a*, and that of the reaction of the plane is *a b*, perpendicular to the plane. Now we *may take* any length along the vertical to represent *the weight* W. *Let a c*, the length taken, be equal to

A C, to which the force is parallel. Through c draw $c\,b$, parallel to F; then $c\,b$ represents F, and $a\,b$ represents R. Now the triangle $a\,b\,c$ has its angles respectively equal to those of the triangle A B C, and a side $a\,c$ equal to A C. Therefore the triangles are equal in every respect; but by the triangle of forces, since there is equilibrium,

$$\frac{\mathrm{F}}{\mathrm{W}} = \frac{c\,b}{a\,c} = \frac{\mathrm{C\,B}}{\mathrm{A\,C}} = \frac{\textit{height}}{\textit{base}}$$

$$\frac{\mathrm{R}}{\mathrm{W}} = \frac{a\,b}{a\,c} = \frac{\mathrm{A\,B}}{\mathrm{A\,C}} = \frac{\textit{length}}{\textit{base}}$$

Example.—What weight will be supported by a power of 10 lbs. on an inclined plane having a height of 4·4 feet in a length of 48·3 feet, the power acting horizontally? Here B C = 4·4, A B = 48·5, A C is found from the formula

$$\mathrm{A\,B}^2 = \mathrm{A\,C}^2 + \mathrm{C\,B}^2$$

$$\therefore\ \mathrm{A\,C} = \sqrt{(48{\cdot}5^2 - 4{\cdot}4^2)}$$

$$= 48{\cdot}3 \text{ feet}$$

$$\text{Now}\quad \frac{\mathrm{P}}{\mathrm{W}} = \frac{\textit{height}}{\textit{base}}$$

$$\therefore\ \frac{10}{\mathrm{W}} = \frac{4{\cdot}4}{48{\cdot}3}$$

$$\mathrm{W} = 483 \div 4{\cdot}4 = 109{\cdot}7 \text{ lbs.}$$

In all these cases the plane is supposed to be fixed. In the second case, instead of the plane being fixed and the body movable, the plane may be movable horizontally and the body vertically. In this case the horizontal force must be applied to the plane

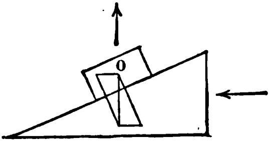


Fig. 84.

(Fig. 84) instead of to the body. The condition of equilibrium remains the same, namely, that

$$\frac{F}{W} = \frac{\text{height}}{\text{base.}}$$

The Wedge.

89. A wedge is a triangular prism used as a movable inclined plane, and is employed to separate bodies that are urged together by great pressures (Fig. 85). The resistances to be overcome act perpendicularly to the faces in contact with the body, and the force is usually applied perpendicularly to the base; hence a section of the wedge perpendicular to the faces sustaining the pressure is a triangle, the sides of which are perpendicular to the forces, and therefore, when there is equilibrium, proportional to the forces. Let P be the pressure necessary to keep the wedge in its place, R the pressure on each face, b the breadth of the base of the triangular section, l the length of one of the equal sides; the equation of equilibrium is therefore

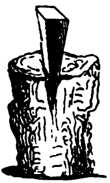


Fig. 85.

$$\frac{P}{R} = \frac{b}{l}$$

In practice, however, the friction between the surfaces plays such an important part, and is so large compared with the power, that the above proportion cannot be applied with any degree of accuracy. Again, the power usually employed with the wedge is not pressure but percussion, and we cannot accurately state the relation between the force of a blow and the resistance it overcomes. Nevertheless one fact shown to be true when friction is neglected is also found to be true in practice, viz. that the more acute the angle of the wedge the more powerful is the instrument.

The Screw.

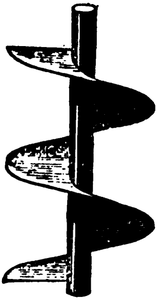


Fig. 86.

90. When an engineer wishes to make a road to the top of a very steep hill, he carries the road round the hill, making it ascend gradually. In a similar manner a path may be made from the bottom to the top of a column by a winding shelf or ledge. Such an arrangement would be a screw (Fig. 86).

A screw may be considered as an inclined plane wound round a cylinder.

The projecting coils are termed the *threads* of the screw. They may be square as in Fig. 88, or triangular as in Fig. 87.

The distance between the upper edge of one thread and the corresponding edge of the next, measured on a line parallel to the axis, is termed the *distance between the threads.* The screw is usually connected with a concave cylinder termed a nut, on the interior surface of which a spiral cavity is cut, corresponding exactly to the thread of the screw which moves in it.

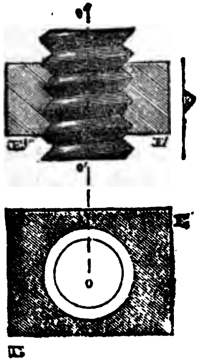


Fig. 87.

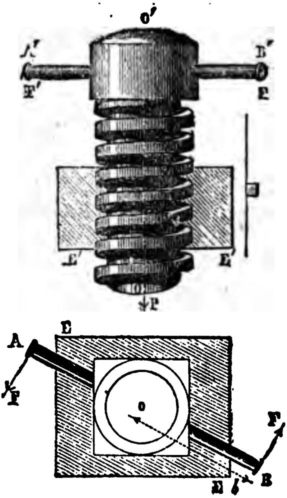


Fig. 88.

The Condition of Equilibrium when a weight is supported by a Screw.

91. Let the weight W be supported by a force P, applied at the circumference of a screw in a direction perpendicular to a plane through the axis. The arrangement is equivalent to an inclined plane acted on by a horizontal force (Fig. 84). The condition of equilibrium is in this case $\frac{P}{W} = \frac{\text{height}}{\text{base.}}$

Now suppose the screw to be unrolled from the cylinder. The whole inclined plane thus formed will be similar to that portion of it which would go exactly once round *the cylinder.* Let A B C be such a portion (Fig. 90).

Then A C is the circumference of the cylinder, B C the distance between the threads. Let A C = C, B C = d

$$\text{then } \frac{P}{W} = \frac{d}{c}$$

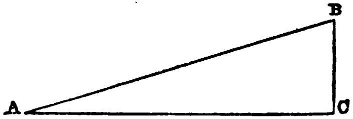


Fig. 89.

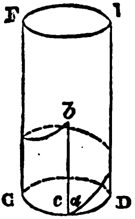


Fig. 90.

The screw is, however, rarely used without a lever. Let F be a power applied at the end of a lever, the length of which from the axis of the cylinder is l, and let r be the radius of the cylinder. Let P be a force applied at the circumference of the cylinder, which would have the same effect as F, then by the principle of the lever, $P \cdot r = F \cdot l$

$$\text{or } P = F \cdot \frac{l}{r}$$

Substitute for P in the above equation, and remark that $C = 2\pi r$, therefore

$$\frac{Fl}{Wr} = \frac{d}{c} = \frac{d}{2\pi r} \text{ hence } \frac{F}{W} = \frac{d}{2\pi l} .$$

or the power is to the weight as the distance between the threads is to the circumference of the circle described by the power.

We may, therefore, increase the mechanical force of the screw, either by diminishing the distance between the threads or by lengthening the arm of the lever, or *by both.*

The screw is commonly used to exert pressure. The press represented in Fig. 91 is a familiar example. The upper beam supports the nut which is fixed. The head C of the screw, moving in a socket fixed to the block D, is pierced for the insertion of levers.

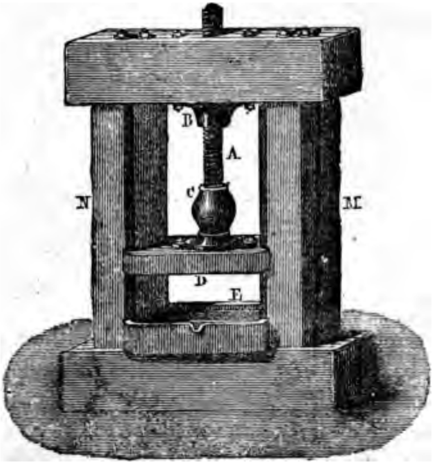


Fig. 91.

The screw is sometimes associated with a spur wheel, and is then termed an *endless* screw, because the teeth of the wheel succeed and replace each other as they advance, so that they never arrive at the end of the *screw*.

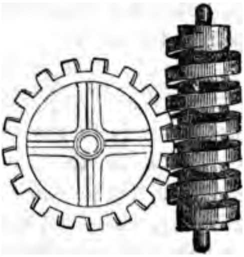


Fig. 92.

EXERCISES.

1. If the power represented by P act in a direction parallel to the plane, and support a weight W, solve the following:—

   1. W = 100 lbs., height = 3 ft., length = 4 ft.; find P.—*Ans.* 75 lbs.
   2. W = 122 lbs., height = 11 in., base = 5 ft.; find R.—*Ans.* 120 lbs.
   3. P = 35 lbs., height = 15 in., length = 27 in.; find W.—*Ans.* 63 lbs.
   4. P = R; find the inclination of the plane.—*Ans.* 45°.
   5. P = 160 lbs., height = 32, base = 255; find W.—*Ans.* 1,285 lbs.
   6. W = 14,267 lbs., height = 72 ft., base = 1,295; find P.—*Ans.* 792 lbs.
   7. A railway train, weighing 50 tons, is supported on an inclined plane, rising 1 ft. for every 50 ft. of length, by a rope attached to a stationary engine; find the tension.—*Ans.* 1 ton.

2. Solve the following exercises, the direction of the power being horizontal:—

   1. W = 15, height = 6, base = 9; find P.—*Ans.* 10.
   2. W = 36, height = 5, length = 13; find P.—*Ans.* 15.
   3. P = 3·5, base = 24, length = 25; find W.—*Ans.* 12.
   4. W = 9, height = 28, base = 45; find R.—*Ans.* 10·6.
   5. $P = \frac{1}{2} R$; find the inclination.—*Ans.* 30°.
   6. P acting parallel to the plane, or 2 P acting horizontally, will support W; find the inclination.—*Ans.* 60°.

3. A weight of 253 lbs. is supported on an inclined plane rising 88 ft. in a length of 137 ft. by two equal forces, one being horizontal and the other parallel to the plane. Find the forces.—*Ans.* 92 lbs.

(A horizontal force P will support P × base ÷ height)

A force P parallel to the plane will support P × length ÷ height

$$\therefore P\frac{105+137}{88} = 253$$

4. A vertical force of 20 lbs. and a horizontal force of 84 lbs.

support a weight on an inclined plane, the height of which is 21 and length 221. Find the weight.—*Ans.* 900 lbs.

5. A weight of 80 lbs. is supported on a smooth inclined plane, the inclination of which is 30°, by a string attached to a point in the plane. Find the tension of the string.

6. What horizontal force will support 100 lbs. on a plane inclined at an angle of 45°?

7. The diameter of a screw is 7 in., and the distance between the threads one-fourth of an inch; what power applied at the circumference of the screw will support a weight of 110 lbs.?—*Ans.* 1·25 lbs.

8. When the circumference of the screw is 12 in., and there are three threads to the inch, find the weight which will be supported by a power of 10 lbs.—*Ans.* 360 lbs.

9. In a common press the diameter of the screw is 6 in., the distance between the threads $\frac{3}{5}$ in., and the length of the lever, measured from the axis, is 4 ft.; what power will support a resistance of 352 lbs.?—*Ans.* ·7 lbs.

10. If the circumference described by the end of the lever be 10 ft., the power 10 lbs., and there be three threads in 2 in., find the resistance supported.—*Ans.* 1,800 lbs.

11. If a screw be formed upon a cylinder whose length is 10 in. and circumference 4 in., how many turns must be given to the thread, in order that the power may be one-eighth of the weight? —*Ans.* 20.

Work done by a Machine.

92. When a weight is lifted through a certain height, the product of the weight in pounds and the height in feet is a measure of the work expended. The work expended in lifting 10 lbs. to a height of 5 ft. is represented by the number 50.

Now the following proposition is true of all machines if we neglect friction:—*The work done by the power is equal to the work done by the weight.* If therefore we imagine the weight to be lifted to a certain height h, and then find the space, p, through which the power will move, the equation $W h = P p$ is always true. We will now apply this principle to determine the equations of equilibrium for the simple machines.

93. *The Lever.* Let A B be a lever, of which C is the fulcrum (Fig. 93), and let a power P at A balance

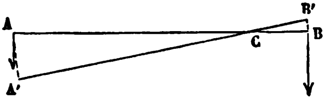


Fig. 93.

a weight Q at B. Suppose the lever to move about the fulcrum, then A will describe an arc A A', and B an arc B B'; then by the above principle $P \cdot AA' = Q \cdot BB'$.

But the spaces A A' B B' are proportional to the arms A C and B C, therefore the lengths of the arms may be substituted for the spaces, thus

$$P \cdot AC = Q \cdot BC$$

94. *The Wheel and Axle.* Let C be the circum-

ference of the wheel, and c that of the axle. Imagine the wheel to go round once, then evidently P will descend through a space C, and Q will ascend through a space c, and by equating the work done we have

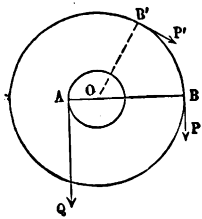


Fig. 94.

$$Q{\cdot}c = P{\cdot}C.$$

Since the radii of circles are proportional to the circumference, we may substitute A O, B O for c and C, and then we have

$$Q{\cdot}AO = P{\cdot}BO$$

95. *The Pulley.* Let us first consider the single movable pulley with parallel strings. Suppose the weight to rise through a height h, then the string on the right will rise through a height h in consequence of the rise of the pulley, and also through a height h in consequence of the shortening of the string on the left. Therefore the point of application of the power will move through a space $2h$: hence

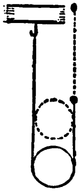


Fig. 95.

$$W{\cdot}h = P{\cdot}2h$$

$$\text{or } W = 2P$$

In the arrangement represented in Fig. 78, if the weight be supposed to ascend 1 foot, it may be shown, by repeating the above reasoning, that the second pulley will ascend 2 feet, the third pulley 4 feet, and nth pulley reckoning from the lowest 2^{n-1} ft., and therefore the power will move through 2^n feet: hence

$$W = 2^n \cdot P$$

In the arrangements represented in Figs. 79 and

80, if the weight ascend through h feet, each of the strings attached to the weight will be shortened h feet, and consequently, if there be 6 parts of the string, the power will move through $6h$ feet

$$\therefore W \cdot h = P \cdot 6h$$
$$W = 6\,P$$

Similarly, if there be n parts of the string to the lower block

$$W = n\,P$$

96. *The Inclined Plane.* First, suppose the force to act in a direction parallel to the plane. Imagine the body to be moved from A to B, then the weight will have been lifted through a height B C, and the power F will have acted in the line of its direction through a space A B. Therefore, by equating the work done,

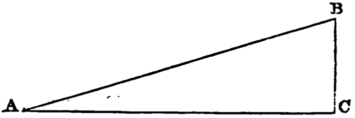


Fig. 96.

$$W \cdot BC = F \cdot AB.$$

This result is the same as that obtained in § 88. The ratio of the power to the weight in this case of the inclined plane was first found by Stevens in the sixteenth century, by the fact that when a chain passes over the top of a smooth inclined plane, so that a part rests on the plane and a part hangs vertically, there will be equilibrium if the two extremities are in the same horizontal line.

Secondly, let us suppose the force to be horizontal. Imagine, as before, that the body moves from A to B, then the weight will have been lifted through a height B C, and the power will have acted in the line of its direction, that is to say, horizontally through a space equal to A C. Hence, by equating the work done, we have

$$W \cdot BC = F \cdot AC.$$

97. *The Screw.* Let the circumference described by the power be C, and let the distance between two threads be *d*. Imagine the power P to act at the end of a lever and to move once round. The screw will rise through a space equal to the distance between two threads, consequently, by equating the work done by power and weight, we have

$$P \cdot C = W \cdot d.$$

Exercises.

Solve the following problems by equating the work done by the power and that done on the weight.

1. A wheel and axle is used to support a weight of 130 lbs. the circumference of the wheel is 8 ft. and that of the axle 6 in.; find the power.

2. Sketch a system of pulleys by which a power, by acting through 64 ft., would raise a weight through one foot. If the power in this case is 10 lbs.: find the weight.

3. In the system of pulleys represented in Fig. 81, if the weight of A be 3 lbs., and that of B be 4 lbs., find the work done by the pulleys when the weight is raised 2 ft.

4. In the preceding example find the weight when the power is 20 lbs., the weights of the pulleys being taken into account.

5. The height of a plane is 5·2 ft. and base 67·5 ft.; what weight will be supported by a power of 13 lbs. acting along the plane?—*Ans.* 169·25 lbs.

6. When a power of 20 lbs. is applied to lift a weight through 2 in. the power descends through 3 in.; find the weight.—*Ans.* 30 lbs.

7. With a wheel and axle a power of 14 lbs. balances 1 ton, if

the power descended through 80 in., through what height would the weight be raised?—*Ans.* ½ in.

8. When there are four movable pulleys in one block, how much string passes through the hands in raising a weight 6 in.? —*Ans.* 4 ft.

9. With a block of three movable pulleys, how much cord would be required for a man to raise himself 30 ft.?—*Ans.* 210 ft.

10. Show that, in order that the pulleys in Fig. 79 may revolve in the same time, the diameters of the lower block must be as the numbers 1, 3, 5, and those of the upper as the numbers 2, 4, 6.

11. The circumference of the circle described by the end of the lever of a screw is 9 ft., and there are two threads to the inch; what power will support 144 lbs. with such a screw?

12. The length of an inclined plain is to the height as 85 to 13; if the power be horizontal and equal to 52 lbs., find the weight.—*Ans.* 336 lbs.

DYNAMICS.

I.—MOTION.

1. The examples of motion which constantly recur to our observation are very varied. Sometimes a body moves in a straight line, as when it falls freely to the earth. Sometimes it describes a curve, as when it is projected in a direction not vertical. Sometimes the body turns and retraces its path, as is the case with the ball of a pendulum. The velocity with which the body moves, and the nature of the forces which produce motion, are other elements admitting of endless variation. It will evidently be necessary, therefore, to consider some of these circumstances of motion apart from the others. For example, it will be convenient to consider, first, the conditions of motion of a body independently of its size and shape, and, making an abstraction of these properties, to consider the motion of a very small particle of matter or a material point.

2. When a moving point or particle is in motion, *the line containing* all its successive positions is termed

the *path* of the point. The path may be a straight line or a curve.

3. The motion of a point is said to be *uniform* when the point passes over equal spaces in equal times.

In the case of uniform motion the velocity of the moving point is constant, and is measured by the length of path passed over in a unit of time. This length is usually expressed in feet, and the time in seconds. Frequently, however, other units are chosen; thus, a train may proceed with a speed of 40 miles an hour, a ship may sail with a speed of 10 knots an hour. Velocity expressed in other units may, however, be readily reduced to feet per second. For example,

$$1 \text{ mile an hour} = 1\tfrac{7}{15} \text{ ft. per sec.}$$

The velocity is *variable* when the lengths of the path described in equal times are not equal.

4. The length of path described in a certain time divided by the time is termed the *mean* velocity for that portion of the path.

5. Variable velocity at any instant is measured by the mean velocity for an infinitely small space commenced at that instant.

It is the space the body would describe in a unit of time if from that particular instant the velocity remained constant. For example, a horse may travel from one place to another with a variable velocity; but we may say that at a particular instant he is running at a speed of 20 miles an hour. We mean that for a small distance he moves with a speed which, if maintained for an hour, would carry him over 20 miles.

Uniform Velocity.

6. Let v ft. per second be the velocity of a moving point. If the velocity be uniform, the point will move through v ft. in every second of time; hence in 2 seconds it will move through 2 v ft., in 3 seconds through 3 v ft., and so on. Consequently, if the point move for t seconds, it will pass through t v ft. Let s = the space passed through in t seconds

$$\text{then } s = Vt.$$

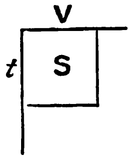


Fig. 97.

7. Suppose two straight lines be drawn at right angles, and v units of length be measured on one and t units on the other, then these lengths may be taken to represent respectively the velocity and the time, and since the area of the rectangle of which these lengths form the sides is represented by the product of the two numbers v and t, this area may be taken to represent the space.

Variable Velocity.

8. When the velocity is variable, the rate of increase or decrease must be known in order that the motion of the point may be determined.

When the velocity varies uniformly, the increase or decrease of velocity per second is termed the *acceleration.*

9. The acceleration is usually denoted by f; so that f = the number of feet added to the velocity in every second of time.

For example, if the velocity of a point at one instant be 30 ft. and a second later 40 ft. per second, then f = 10 ft. per second. If at one instant the velocity be 10 ft., and *three* seconds later 70 ft., the acceleration is 60 ft. ÷ 3, or 20 ft. per second.

10. From the preceding definitions it follows that if v be the velocity at the commencement of the time, the velocity at the end of the

$$\begin{aligned} \text{1st sec.} &= v + f \\ \text{2nd sec.} &= v + 2f \\ \text{3rd sec.} &= v + 3f \\ &\cdots\cdots\cdots \\ \text{at time } t &= v + ft \end{aligned}$$

11. Let us take a vertical line, as in Fig. 98, on which we can represent intervals of time, and at the points representing different instants of time draw horizontal lines proportional to the velocities at these instants, then by joining the extremities of these lines we obtain a figure the area of which represents the space passed over precisely, as in the case of a constant velocity.

12. Let the velocity at the commencement of the motion be 0, then at the end of a second the horizontal line will represent f ft. at the end of 2 seconds $2f$ ft., and so on.

The space passed over in the time t will now be represented by a triangle. Now the area of a triangle is found by multiplying half the base by the height.

$$\therefore \; s = \tfrac{1}{2} f t^2.$$

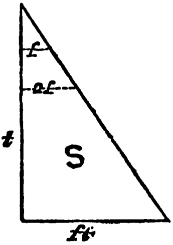


Fig. 98.

13. If the initial velocity be v, the velocity at the end of t seconds will be $v + ft$, and the space will be represented by Fig. 99, which consists of a rectangle, whose area is vt, and a *triangle*, whose area is $\frac{1}{2} f t^2$

$$\therefore \; s = Vt + \tfrac{1}{2} f t^2.$$

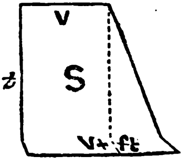


Fig. 99.

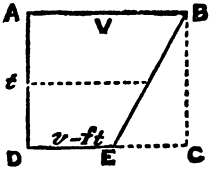


Fig. 100.

14. If the velocity be diminished uniformly by f ft. per second, we have the velocity at the end of t seconds, $v = \text{V} - ft$, and the space s, represented by the rectangle A B C D $(= \text{V}t)$ — the triangle B E C $(= \frac{1}{2}ft^2)$

$$\therefore \; s = \text{V}t - \tfrac{1}{2}ft^2.$$

15. The area of either of these figures may also be found by multiplying the mean breadth (*i.e.* half the sum of the upper and lower breadths) by the height; *hence the space passed over by a particle moving with uniform acceleration is found by multiplying half the sum of the initial and final velocities by the time.*

16. From the two formulæ of motion

$$v = \text{V} \pm ft$$
$$s = \text{V}t \pm \tfrac{1}{2}ft^2$$

we may obtain a third, which does not involve the time, thus:—Multiply the first by t and subtract twice the second,

$$\therefore \; t = \frac{2s}{v \pm \text{V}}$$

But from the first, $t = \dfrac{v \pm \text{V}}{f}$; by equating these values of t, we obtain $v^2 = \text{V}^2 \pm 2fs$.

17. We have now three equations of motion,

$$v = \text{V} \pm ft \qquad (\text{I.})$$
$$s = \text{V}t \pm \tfrac{1}{2}ft^2 \quad (\text{II.})$$
$$v^2 = \text{V}^2 \pm 2fs \quad (\text{III.})$$

Where V is the initial velocity,
f the velocity added in every second,
t the time during which the body has been moving.

EXAMPLES.

1. A point moves from rest with uniform acceleration; the velocity at the end of three seconds is 18 ft.; what is the velocity at the end of seven seconds?

The velocity gained in 3 secs. $=$ 18 ft.
The velocity gained in 1 sec. $= \frac{18}{3} = 6$ ft.
$\therefore$ The velocity gained in 7 secs. $= 7 \times 6 = 42$ ft.

2. A body moves at one instant with a velocity of 30 ft. per second, and 5 seconds later with a velocity of 50 ft. per second; find the acceleration supposing it to be uniform.

Velocity at the end of 5 secs. $=$ 50
Velocity at the beginning $=$ 30
Gain in 5 secs. . . . . . . $=$ 20
$\therefore$ Gain in 1 sec. . . . . . . $=$ 4

3. Find the space passed over in six seconds by a point which moves from rest and acquires a velocity of 30 feet.

First method:—

Initial velocity . . . . $=$ 0
Final velocity . . . . $=$ 30
By adding and dividing by 2 we have the mean velocity . . . $=$ 15
Therefore the space $= 15 \times 6 = 90$

Second method:—

Velocity gained in 6 secs. . $=$ 30 ft.
$\therefore$ The velocity gained in 1 sec. $=$ 5 ft.
Thus $f = 5$. But $S = \frac{1}{2} f t^2$
$\therefore$ $S = \frac{1}{2} \times 5 \times 36 = 90$ ft.

4. If a point start with velocity of 20 ft. per second and gain a velocity of 10 ft. per second, find the space passed through in 8 seconds.

Here $V = 20$, $f = 10$, and $t = 8$
but $S = Vt + \frac{1}{2} f t^2$
$\therefore$ $S = 20 \times 8 + \frac{1}{2} \times 10 \times 64 = 480$ ft.

Or thus :—

Initial velocity . . . . = 20 ft.
Final velocity = $20 + 10 \times 8 = 100$

Adding and dividing by 2 we obtain the mean velocity . . } = 60
Therefore the space = $60 \times 8 = 480$ ft.

5. What is the velocity of a body which commenced to move with a velocity of 80 ft. per second, and has continued in motion for 7 seconds, the velocity regularly diminishing at the rate of 8 ft. per second?

Decrease per second . . . . = 8 ft.
" in 7 seconds . . . = 56
∴ Final velocity = $80 - 56 = 24$.

6. Show by a diagram that the spaces passed through in successive seconds by a point moving from rest under the action of a force producing a uniform acceleration are as the odd numbers 1, 3, 5, &c.

Let O M be the line along which to measure intervals of time. Let O a_1 represent the first second, a_1 a_2 the next, a_2 a_3 the third, and so on. Let a_1 v_1, a_2 v_2, a_3 v_3, &c. represent the velocities after 1, 2, 3 seconds respectively. The figures O a_1 v_1, a_1 v_2, a_2 v_3, &c. represent the spaces passed over in the successive seconds; but by drawing lines through the points v_1 v_2 v_3, &c. parallel to O M, and through a_1 a_2 a_3, &c. parallel to O N, we divide all the figures into triangles equal to O a_1 v_1, the numbers of triangles in the successive figures being respectively 1, 3, 5, 7, &c., therefore the spaces have the proportion of these numbers.

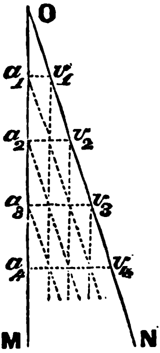


Fig. 101.

7. A particle A starts from a point with a velocity of 20 feet per second accelerated uniformly by 10 ft. a second. Four *seconds later* another particle B starts after A with a velocity of *100 ft. per second,* accelerated regularly by 20 ft. per second.

When B overtakes A, how long will A have moved, and how far will it be from the starting point?

Let t be the number of seconds through which A will have moved, and s the space, then $(t - 4)$ will be the time during which B will have moved, and the space s will be the same. The formula for A's motion will therefore be

$$s = V_1 t + \tfrac{1}{2} f_1 t^2$$

$$\text{or} \quad s = 20t + 5t^2 \qquad \text{(I.)}$$

And the formula for B's motion

$$s = V_2 (t - 4) + \tfrac{1}{2} f_2 (t - 4)^2$$

$$\text{or} \quad s = 100 (t - 4) + 10 (t - 4)^2$$

$$= 10t^2 + 20t - 240 \qquad \text{(II.)}$$

Equate these two values of s,

$$\therefore \quad 20t + 5t^2 = 10t^2 + 20t - 240$$

$$\therefore \quad 5t^2 = 240$$

$$t^2 = 48 = 16 \times 3$$

$$t = 4\sqrt{3}.$$

EXERCISES.

1. A point moves from rest with uniform acceleration, and the velocity at the end of 4½ seconds is 90 feet; find the acceleration.

2. Define the term acceleration. A body moves at one instant with a velocity of 30 feet per second, and 7 seconds later with a velocity of 79 feet per second. Supposing the velocity to be uniformly accelerated, find the acceleration.

3. Explain clearly the meaning of the equation $v = V \pm f t$.

4. Find by the help of a diagram the space passed over in 5 seconds by a body which began to move with a velocity of 17 feet per second, and acquired a velocity 43 feet per second by the end of the fifth second, supposing the acceleration to be uniform.—*Ans.* 150 ft.

5. Explain the meaning of the symbols in the equation

$$s = Vt \pm \tfrac{1}{2} f t^2$$

and show by means of a diagram how it may be obtained.

6. A body starts with a velocity of 80 feet per second, and the velocity decreases regularly by 10 feet a second. What will be the velocity after 7 seconds?

7. In the preceding example what will be the space passed over?—*Ans.* 315 ft.

8. A point starts from rest with an acceleration of 32 feet per second, how far will it move in 8 seconds?—*Ans.* 1,024 feet.

9. The acceleration is 8 ft. per second, and the initial velocity 11 ft. per second; find the space passed over in 8 seconds, and the velocity at the end of the time.—*Ans.* 344 ft.; 75 ft.

10. The space passed over in 5 seconds is 105 ft., and the final velocity 35 ft.; find the initial velocity and the acceleration.—*Ans.* 7 ft.; 5·6 ft.

11. A body moving from rest is observed to move over 6 ft. and 11 ft. respectively in 2 consecutive seconds; show that the acceleration is 5 ft.

12. A body uniformly accelerated moves at the end of 2 seconds with a velocity which would carry it through 5 miles in the next half hour; find the acceleration.—*Ans.* $7 \cdot \dot{3}$ ft.

13. A body moving with uniform acceleration describes 570 ft. in the eighth second; find the acceleration.—*Ans.* 76 ft.

14. A body has described 392 ft. from rest in 7 seconds; find the velocity acquired.—*Ans.* 112 ft.

15. A body has described 54 ft. from rest in 3 seconds; find the time it will take to move over the next 120 ft.—*Ans.* 2·38 seconds.

16. A body moves over 70 ft. in the fourth second; find the acceleration.—*Ans.* 20 ft.

17. A body describes 354 ft. while its velocity increases from 43 to 75 feet per second; find the whole time of motion and the acceleration.—*Ans.* 6 secs.; 5·3 ft.

18. A body in passing over 135 ft. has its velocity increased from 7 to 53; find the whole space described from rest and the acceleration.—*Ans.* $137\frac{73}{184}$ ft.; $10 \cdot \dot{2}$ ft.

19. Find the numerical value of the acceleration when in a quarter of a second a velocity is produced which would carry a body over 2 ft. in every third of a second.—*Ans.* 24.

20. A body moving from rest is observed to pass over 36 ft. and 44 ft. respectively in 2 consecutive seconds; find the acceleration and the time from rest.—*Ans.* 8 ft.; 4 and 5 secs.

II.—FORCES PRODUCING MOTION.

18. Hitherto we have considered the motion of a particle under various circumstances independently of the forces which produce the motion. We have now to consider the relations between the forces and the motion which they produce.

The science of Dynamics rests on certain fundamental principles termed the Laws of Motion.

The first law of motion, or the law of inertia, is as follows:—

When a body is not acted on by any force, if it be at rest it will remain at rest, and if it be in motion it will continue to move in a straight line with uniform velocity.

As this law is the basis of Statics as well as Dymamics it was stated at the introduction, together with the evidence on which it rests.

On this and the following law the theory of the motion of the heavenly bodies is based, and the uniform agreement of the deductions from these laws and observations in astronomy is one of the strongest confirmations of their truth.

19. The second law may be stated thus:—

When a force acts upon a body in motion, the change of motion is the same in magnitude and direction as if the force acted on the body at rest.

The truth of this law is shown by such considerations as the following:—

When a person is on board a boat which is moving uniformly along a stream, any movement he makes produces exactly the same effect as if the boat were at rest.

When a stone is let fall from a point on land it falls in the direction of the vertical, and when a stone is let fall from the mast of a ship in motion, it reaches the deck at a point vertically below the starting point. Now the stone falls from the mast to the deck in the same time whether the vessel be at rest or in motion; again, if the vessel passes horizontally through any distance, 3 ft. suppose, during the fall the stone also passes through 3 ft. horizontally, that is, through the same space as it would have passed through had it remained at the top of the mast. We conclude, therefore, that the horizontal motion due to the velocity of the vessel, and the vertical motion due to the attraction of the earth, have each their full effect in their own direction, that is to say, in the resultant motion the stone is displaced horizontally in a certain time exactly as if its vertical motion did not exist, and it is displaced vertically in the same time as if its horizontal motion did not exist.

This will be made still plainer by reference to the following experiments:—

Take a board A B E (Fig. 102) which has a curved groove A B cut in it, so that the direction of the curve at B is horizontal. If a ball *m* be made to slide down this groove the direction of its motion is horizontal. If the groove be prolonged horizontally to the point M, and the surface of the groove and ball be so well polished that friction may be disregarded, after the ball has passed B it will move along B M with uniform *velocity*. *At the* instant the ball *m* passes the point B, let another ball *m'* fall vertically along B E, and let B M be taken at such a length that *m* moves from B to M in

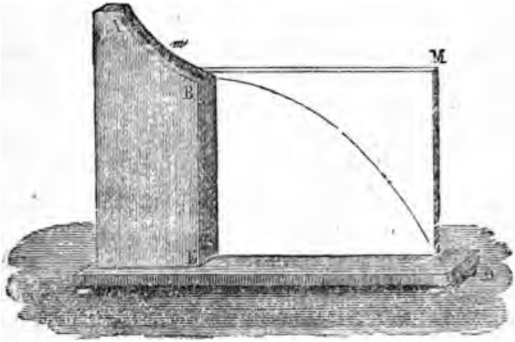


Fig. 102.

exactly the same time as *m'* moves from B to E. If now the horizontal groove be removed, and the ball be allowed to slide down the groove A B, and then to fall freely, it will neither arrive at the point M nor the point E, but at a point D, having passed through a vertical distance B E and a horizontal distance B M. It will arrive at the point D in exactly the same time as the ball *m'* falling free would pass from B to E or the ball *m* after descending through the arc A B would pass from B to M.

Falling Bodies.

20. The fall of bodies to the earth in various circumstances offers remarkable illustrations of the preceding principles.

When bodies of different material fall in the air, they do not usually pass through the same heights in the same time. A ball of lead and a scrap of paper fall through the air with very different velocities. The difference arises from the difference in the resistance offered by the air, which varies with the form and dimensions of the body and with the velocity.

If the bodies are made to fall in a tube from which the air has been expelled, then the time of descent and the velocity acquired will be the same. The motion of all bodies in vacuo is uniformly accelerated.

It is usual to call the force producing the motion "gravity," and to indicate the acceleration by g. Hence g is the number of feet added to the velocity of a body moving freely in vacuo for every second of time. This acceleration is not absolutely the same at all points on the earth's surface, but its variations are very small. It increases with the latitude of the place and decreases with the height above the sea. In London a velocity of nearly 32·2 feet is added in every second when a body moves in vacuo.

We may therefore at once apply the formulæ for uniform acceleration to the case of falling bodies. When the body falls from rest the equations of motion (§ 15) are

$$v = gt \qquad s = \tfrac{1}{2} gt^2$$

and by equating values of t from both equations $v^2 = 2gs$.

When the body has an initial velocity v these equations of motion are

$$v = V \pm gt \qquad S = Vt \pm \tfrac{1}{2} gt$$

and by finding the value of t from the first equation and substituting in the second

$$v^2 = V^2 \pm 2gs.$$

In all cases in which the body moves in a vertical line the space passed through is equal to the mean velocity multiplied by the time.

Example 1. A body falling under the action of gravity has a velocity of 30 ft.; how far will it fall in 5 seconds?

Initial velocity = 30 ft.
Final ,, = $30 + 5 \times 32$
Mean ,, = 110
5
Space ,, 550 ft.

Example 2. A body is projected vertically upwards with a velocity of 100 feet per second; how far will it ascend in 3 seconds?

Initial velocity = 100
Final ,, = $100 - 3 \times 32$
Mean ,, = 52
3
Space ,, 156

Example 3. To find the space passed through in the n^{th} second:—

Velocity at commencement of n^{th} second $= \overline{n - 1} \cdot g$.

Velocity at end of n^{th} second $= n\,g$.

Mean velocity and space $= \frac{1}{2}(2\,n - 1)\,g$.

Substituting successively 1, 2, 3, &c. for n, we find the spaces passed through in the successive seconds are $\frac{g}{2}$, $3\,\frac{g}{2}$, $5\,\frac{g}{2}$, $7 \cdot \frac{g}{2}$, &c., and generally the space passed through in the n^{th} second is the n^{th} odd number multiplied by $\frac{g}{2}$.

Fig. 103 will therefore represent the divisions of the path corresponding to different times, each part representing $\frac{g}{2}$ feet.

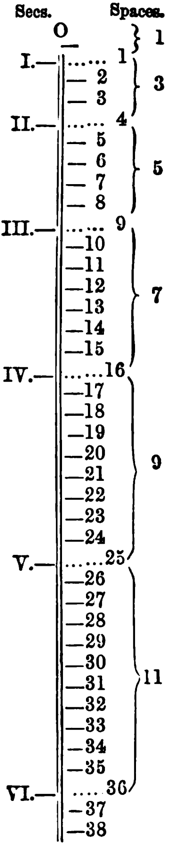


Fig. 103.

Example 4. A body is projected vertically upwards with velocity v; to what height will it rise?

The velocity at any point is given by the equation $v^2 = \mathrm{v}^2 - 2gs$; but at the highest point the velocity is o. Hence, if h be the height $0 = \mathrm{v}^2 - 2gh$,

$$\text{hence } h = \frac{\mathrm{V}^2}{2g}$$

The velocity acquired in falling from this height is given by the formula $v^2 = 2gh$; hence $v = \mathrm{v}$, and the body strikes the ground with a velocity equal to the velocity of projection.

Example 5. Find the time of ascent.

Here $\mathrm{s} = \mathrm{v}t - \frac{1}{2}gt^2$

Put $\mathrm{s} = 0$, then $\mathrm{v}t - \frac{1}{2}gt^2 = 0$,

and $t = 0$, or $\dfrac{2\mathrm{V}}{g}$

The first value corresponds with the instant of starting, and the second with the instant at which the body returns to the starting point.

Again, put $\mathrm{S} = \dfrac{\mathrm{V}^2}{g}$, that is, the height to which the body will rise,

$$\text{then } t = \frac{\mathrm{V}}{g}$$

Now, this is half the whole time; hence the time of ascent is equal to the time of descent.

These results are important, and may be collected thus:—

When a body is projected vertically upwards with velocity v, *it rises for* $\dfrac{\mathrm{V}}{g}$ *seconds, reaches the height*

$\frac{V^2}{2g}$, *falls in the same time as it took to ascend, and strikes the ground with the velocity* V.

21. Let us now apply the second law of motion to find the position of a particle projected horizontally with velocity v, and allowed to fall under the action of its weight, as represented in Fig. 102.

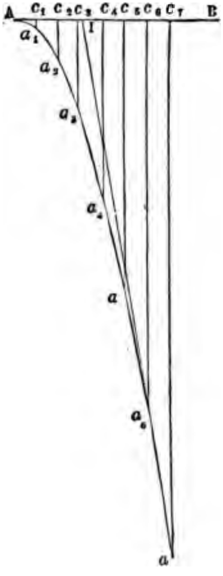


Fig. 104.

Let A B represent the horizontal direction, and, according to a fixed scale, take equal distances A C, $C_1 C_2$, &c. to represent the spaces which would be passed over in successive seconds if the body were subject only to the force of projection. At the points C_1, C_2, &c. draw vertical lines representing, according to the scale chosen, the spaces which would be passed through in 1, 2, 3, &c. seconds if gravity were the only force. Thus, $C_1 a_1 =$ 16 ft., $C_2 a_2 =$ 16 × 4 ft., $C_3 a_3 =$ 16 × 9 ft., &c. The points a_1, a_2, a_3, &c. represent the position of the particle at the end of the successive seconds, and the curve joining these points represents the path.

EXERCISES.

In the following exercises use 32 ft. as an approximation for the acceleration due to gravity.

On the formula $v = \mathrm{V} \pm g\,t$.

1. A body falls for 8 seconds; with what velocity is it moving at the end of that time?—*Ans.* 256.

2. If a body is let fall, how long will it take to acquire a velocity of 500 ft. per second?—*Ans.* $15\frac{5}{8}$ secs.

3. A body is projected upward with a velocity of 80 ft. per second; determine the velocity it will have at the end of 3 seconds, and the number of seconds that must elapse before its velocity equals its initial velocity.—*Ans.* 16 ft.; 5 secs.

4. A body is thrown downward with a velocity of 160 ft. per second; find its velocity at the end of 5 seconds, and the number of seconds in which a body that is merely dropped would acquire that velocity.—*Ans.* 320; 10 secs.

5. A body A is projected downward with a velocity of 240 ft. per second; at the same instant another body B is projected upward with an equal velocity; how much faster will A be moving than B at the end of 6 seconds?—*Ans.* 9 times.

6. A body is thrown upward with a velocity of 96 ft. per second; with what velocity will it be moving at the end of 4 seconds?—*Ans.* Downwards with a velocity of 32 ft. per second.

7. A body is dropped from a certain height h, at the same instant as another is thrown upward; what initial velocity must the latter have that it may meet the former half way?—*Ans.* $\sqrt{gh}$.

8. A body is at a given instant moving upward with a given velocity v; show that it will be moving downward with an equal velocity after $\frac{2\,v}{g}$ seconds, and that it will reach its highest point after $\frac{v}{g}$ seconds.

9. A body is thrown up with a velocity $a\ g$; after how long will it be descending with a velocity $b\ g$?—*Ans.* $a + b$.

On the formula $S = Vt \pm \frac{1}{2} g t^2$

10. How many feet will be described in 7 seconds by a body that moves freely from rest under the action of gravity?—*Ans.* 784.

11. Through how many yards would a body falling freely from rest descend in 3 minutes?—*Ans.* 172,800.

12. A body is projected downward with a velocity of 50 ft. per second; how far will it fall in $4\frac{1}{2}$ seconds?—*Ans.* 549 ft.

13. A body is projected upward with a velocity of 150 ft. per second; how high will it have ascended in $6\frac{1}{4}$ seconds?—*Ans.* 299 ft.

14. If a body is thrown upward with a velocity of 96 ft. per second, how far will it be from the starting point at the end of $4\frac{3}{4}$ seconds, and what will be the whole space it will have described?—*Ans.* 95 ft.; 193 ft.

15. A body is projected upward with a velocity of $3g$ ft. per second; determine the height of the body, and with what velocity, and in what direction, it will be moving at the end of 4 seconds.—*Ans.* Height $4g$, moving downward with velocity g.

16. A body is projected upward with a velocity v; show that it will return to the point of projection after $\frac{2v}{g}$ seconds.

17. A body falls for a time t, and has a velocity V at the beginning, and v at the end of that time; find the space described. —*Ans.* $\frac{1}{2}(V + v)\ t$.

On the formula $v^2 = V^2 \pm 2 g s$.

18. If a body is thrown upward with a velocity of 36 ft. per second, find its greatest height.—*Ans.* $20\frac{1}{4}$ ft.

19. If a body falls freely through 1,600 ft., find the velocity it acquires.—*Ans.* 320 ft.

20. A body is projected vertically upward with a velocity of 100 ft. per second; how long will it take to reach the top of a tower 100 ft. high?—*Ans.* $1\frac{1}{4}$ or 5 secs.

21. A body is thrown upwards with a velocity of 80 ft. per second; how far will it ascend?

22. Two stones are dropped from different heights and reach the ground at the same time, the first from a height of 81 feet and the second from a height of 49 feet; find the interval between their starting.—*Ans.* $\frac{1}{2}$ sec.

23. A body is projected upwards with velocity of 60 feet per second; what will be its velocity after it has passed over 50 feet?—*Ans.* 20 ft.

24. A body is thrown downwards with a velocity of 10 feet per second; what will be its velocity after it has fallen through 75 feet?

25. A stone A is let fall from a certain point, and after it has fallen for a second, another stone B is let fall from a point 100 ft. lower down; in how many seconds will A overtake B?—*Ans.* $3\frac{5}{8}$ seconds from the starting of A.

26. With what initial velocity must a body be thrown downwards that it may strike the ground, which is 69 ft. below the starting point, with a velocity of 104 ft. per second?—*Ans.* 80 ft.

27. A body commences to move with a velocity of 80 ft. per second, and after it has passed over 100 ft. it has a velocity of 120 ft. per second; find the acceleration.

Mass.

22. Hitherto we have considered a force producing motion as measured by the acceleration or the velocity it generates. It is, however, necessary to take into account the quantity of matter moved. When the same force acts on different bodies, it produces in each a

uniform acceleration; but the acceleration will not be the same in both cases.

This difference arises from what is called the mass of the bodies.

23. *Mass* is a term for the quantity of matter in a body. It is not easy to give such a definition of mass as will lead us at once to a method of measuring it. We are compelled to measure mass by its effects.

We assume that the attraction of the earth on all particles of matter is the same, and is not dependent on the nature of the matter attracted. This assumption seems to be justified by the fact that bodies of all kinds fall with equal velocity in the exhausted receiver of an air-pump. Hence we measure the mass of a body by its weight, and can only define the mass as a quantity proportional to the weight. If, then, at the same spot in the earth's surface one body is twice as heavy as another, the mass of the first is twice that of the second.

Suppose, however, the body is weighed by a spring balance (Fig. 2) at a certain place, and weighed again by the same instrument at another place nearer the equator, it will be found that the body is lighter at the latter place. It is found also that the acceleration due to the attraction of the earth is also less at the second place than at the first in the same proportion. This illustrates the fact that when the mass remains the same, the weight varies as the acceleration due to gravity. Consequently, writing M for mass, W for weight, and g for acceleration due to gravity,

W varies as M when g is constant,
W varies as g when M is constant;

therefore W varies as Mg when both vary; and therefore, by a theorem of algebra, if

$$C = \text{some constant quantity,}$$

$$W = CMg.$$

By selecting a suitable unit of mass we may make this constant unity;

$$\text{then } W = Mg.$$

This formula exhibits a relation between the statical and the dynamical measure of force. When a force acts upon a particle, the dynamical effect of the force (supposing the particle to move) is measured by the acceleration produced. The statical effect, supposing the particle to be kept at rest, is the pressure produced; and if P be the pressure, M the mass, and f the acceleration, then $P = M \cdot f$.

If for M we write its value in the above equation, we have

$$f = g \frac{P}{W}$$

This result may be expressed in words as follows:—

24. *When pressure produces motion, the acceleration varies directly as the pressure, and inversely as the mass.*

The product Mf is sometimes called the *moving force*, and this statement is then made thus—the moving force is proportional to the pressure. This law may be illustrated by the following arrangement:—

Suppose two weights P and Q, whose masses are respectively m and m', to be connected by a fine inextensible cord passing over a fixed pulley. If $P = Q$ there will be no motion; if P be greater than Q there will be a pressure producing motion equal to the excess of P over Q or (P — Q). The mass moved is $m + m'$, hence

$$f(m + m') = P - Q,$$

$$\text{but } m = \frac{P}{g} \text{ and } m' = \frac{Q}{g}$$

$$\text{therefore } f = g\,\frac{P - Q}{P + Q}$$

25. *To find the tension in the cord.*

We may consider the tension T to be the upward force acting both on P and Q; the pressure producing the motion of P downwards will be P — T, and the pressure moving Q upwards T — Q, and the accelerations are the same;

$$\text{hence } f = \frac{P - T}{m}$$

$$\text{and } f = \frac{T - Q}{m'};$$

$$\text{therefore } \frac{P - T}{m} = \frac{T - Q}{m'}$$

or dividing by g,

$$\frac{P - T}{P} = \frac{T - Q}{Q}$$

$$\text{consequently } T = \frac{2PQ}{P + Q}$$

For example, suppose two weights connected by a cord passing over a fixed pulley to be 3 ozs. and 5 ozs. respectively; then, $f = 32 \cdot \frac{2}{8} = 8$, and if the weights are allowed to move from rest, the equations of motion become $v = 8t$ and $s = 4t^2$. The tension in the cord will be $\frac{2 \times 3 \times 5}{3 + 5} = 3\frac{3}{4}$. If the 8 lbs.

instead of being divided into unequal parts 3 and 5 were divided into equal parts, the tension of the string would be 4 lbs.; hence we see that the tension is less when the parts are unequal than when they are equal. That which is true in this particular case is generally true.

Attwood's Machine.

26. The machine represented in the figure was devised by Attwood for testing experimentally the laws of motion and the results derived from the theory of falling bodies.

It consists of an upright beam, supporting at the upper extremity a nicely constructed wheel turning on a horizontal axis, and two equal weights P, P′, connected by a fine silk thread which passes over a groove in the wheel. To diminish friction the axis of the larger wheel turns on friction wheels (Fig. 105).

The pillar is furnished with a graduated scale, a movable ring *n*, and stage *m*.

Another important appendage to the machine is a small clock with a pendulum beating seconds.

Suppose the weights P and P′ to be at first equal: they will then be at rest.

Let the movable ring be in the position *n*, below the weight P and on this weight place a small bar *p*. P will descend with accelerated velocity until it reaches the ring (*n*). Here the bar *p* will be lifted off the weight P by means of the ring (*n*), and P will move onward with a uniform velocity equal to the velocity at the moment when (*p*) was retained by the ring. Now place *the stage* in such a position that the weight P

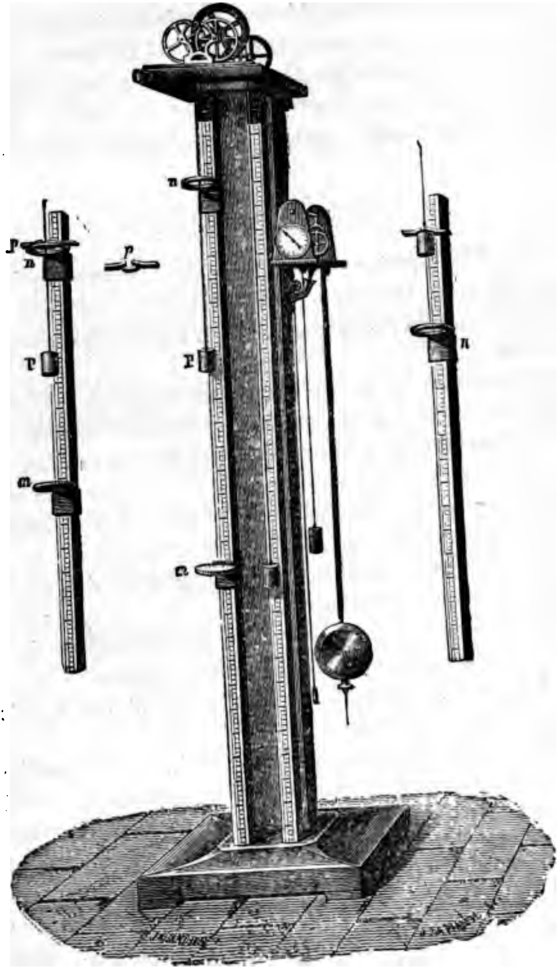


Fig. 105.

may reach it in exactly one second after it has passed the ring (n). The distance between the stage and ring will be the space described by a body moving with a uniform velocity in a unit of time.

Let us now suppose that the ring is exactly so far below the starting-point P as to lift off the bar when P has moved for exactly one second. Also let the stage be so placed as to stop P exactly one second after the bar has been removed. Then the distance n m is the acceleration. Now the following relation is found to exist between this acceleration and the weights: if we double the weight of the bar p, while we keep the sum of the weights P P′ and p the same, we double the acceleration, and if we take a bar n times the weight of the first, the second acceleration will be n times that of the first.

If now we keep the bar the same, but double the sum of the weights, we shall find that the acceleration is then only half what it was before, and generally if we multiply the weight moved by n we shall divide the acceleration by n.

Thus the acceleration varies directly as the pressure producing motion, and inversely as the mass moved. In every respect we find the results agree with the theory of Article 33.

The chief advantage secured by this machine is that we may make the acceleration as small as we please, and thus render the motion slow enough to be observed without difficulty.

Suppose, for example, that the equal weights are each 31·7 grammes and the bar 1 gramme, then, taking the acceleration due to gravity as 32·2, we have, by the theory of Article 33, an acceleration equal to $32{\cdot}2 \times \frac{1}{32{\cdot}7 + 31{\cdot}7} = \frac{1}{2}$. Now if we keep the bar on throughout the time it is found that the space described in the first second is $\frac{1}{4}$ ft., in the next second $3 \times \frac{1}{4}$ ft. in the third $5 \times \frac{1}{4}$ ft. and in the nth second the nth odd number multiplied by $\frac{1}{4}$ ft.

Again, by cutting off the pressure producing motion at the end of the successive seconds, and measuring the spaces passed through in the next second, it is found that the velocity at the end of the first second is $\frac{1}{2}$ ft., at the end of the next $2 \times \frac{1}{2}$ ft., and at the end of the third $3 \times \frac{1}{2}$ ft., and at the end of the nth second $n \times \frac{1}{2}$ ft. Thus the results of the experiment agree entirely with those of the theory of falling bodies.

EXAMPLES.

1. What velocity will be communicated to a weight of 100 lbs. by a pressure of 10 lbs. in 10 seconds?

From the formula $f = g\frac{P}{W}$ since $g = 32$, $P = 10$ and $W = 100$ lbs., we have $f = 32 \times \frac{10}{100} = 3{\cdot}2$; and from the formula $v = ft$ we have $v = 3{\cdot}2 \times 10 = 32$.

2. A weight of 8 lbs. hanging over the edge of a smooth table pulls along a weight of 28 lbs. placed on the table; find the acceleration and the space passed over in 3 secs.

Here the pressure producing motion is the 8 lbs. and the weight moved $8 + 28 = 36$ lbs.

$$\therefore\ f = 32 \times \frac{8}{36} = 7{\cdot}\dot{1}$$

Again, since $s = \frac{1}{2}ft^2$

if $t = 3$ secs., we have $s = \frac{1}{2} \times 7{\cdot}\dot{1} \times 9 = 32$.

EXERCISES.

1. A pressure of 2 lbs. moves a weight of 12 lbs.; find the acceleration.

2. How far will a weight of 50 lbs. be moved by a pressure of 1 lb. in 12 seconds?

3. What pressure by acting for 6 seconds will produce a velocity of 120 feet per second in a weight of 12 lbs.?—*Ans.* 7½ lbs.

4. If weights of 3 and 5 lbs. are connected by a string which passes over a fixed pulley, what will be the velocity generated in a second of time?

5. In the above example, how far will the heavier weight descend in 5 seconds?—*Ans.* 100 ft.

6. If the weights be respectively 15 lbs. and 17 lbs., in how long will they move through 144 feet?

7. A weight of 2 lbs. hangs over the edge of a smooth table, and draws a weight of 50 lbs. laid on the table; what velocity will be acquired in $2\frac{1}{2}$ seconds?—*Ans.* $3\frac{1}{13}$ feet.

8. How far will either of the weights move in 5 seconds?—*Ans.* $15\frac{5}{13}$ feet.

9. A steam-engine moves a train weighing 60 tons on a level road from rest, and acquires a speed of 5 miles an hour in 5 minutes. If the same engine move another train, and give it a speed of 7 miles in 10 minutes, find the weight of the second train, supposing the resistance to amount to the same in both cases?—*Ans.* $85\frac{5}{7}$ tons.

MOMENTUM.

27. We have already shown that if P be a pressure producing the motion of a mass M, and if f be the velocity generated in a unit of time, then by taking a suitable unit of mass the relation of these quantities is expressed by the equation $P = fM$.

The product of the mass into the velocity at any instant is termed the *momentum* at that instant.

Hence, when the acceleration is uniform, fM is the momentum at the end of the first second, and also the increase of momentum for every second. The above equation states that *the momentum* generated in a unit of time is proportional to the force.

The momentum bears the same relation to the moving force that the velocity does to the acceleration; the momentum varies with the time, and the moving force *is the* increase of momentum for a second of time.

28. When a moving body produces the motion of another body, the greater the momentum of the first the greater the momentum generated in the second.

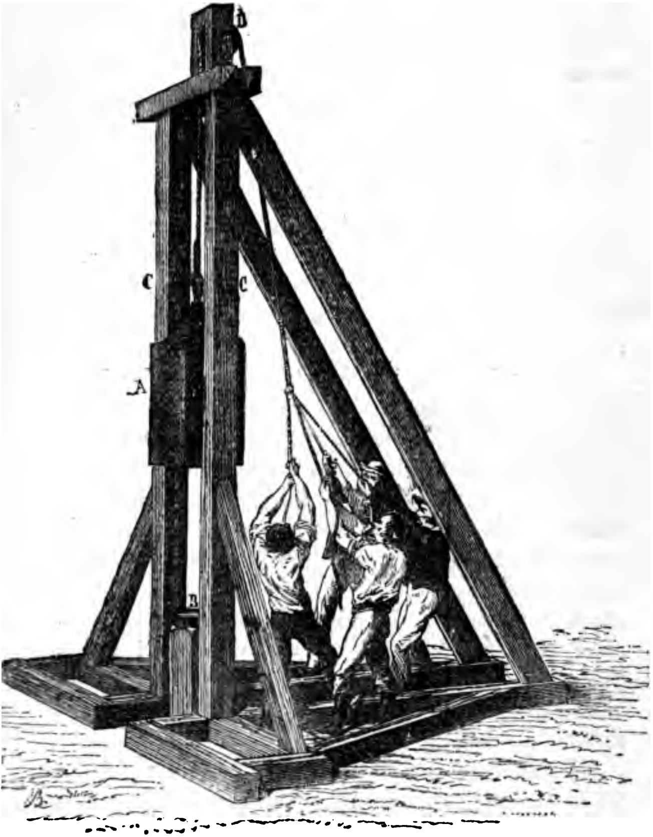


Fig. 106.

As an illustration, refer to the machine for driving piles, represented in Fig. 106. The block of iron A is raised to the top of the frame C C, and allowed to fall. It might lie on the head of the pile B for an indefinite length of time without producing any effect, but a momentum is generated in the fall which is destroyed in an exceedingly short time by the resistance of the pile. On the one hand we see that the greater the mass A, and the greater the height from which it falls, the greater is the effect, and on the other hand the greater the resistance of the pile the shorter the time during which it is overcome by the momentum of A.

Whenever the resistance to be overcome is great, the action of the force must be limited to a proportionately short time.

When a hammer is made to fall on the head of a nail the momentum of the hammer at the instant of contact is spent in overcoming the cohesion of the particles of the wood. With a given momentum the shorter the duration of the shock the greater the effect of the force in overcoming this resistance. Suppose that the board is not firmly supported so that it yields under the blow (Fig. 133), the duration of the shock is prolonged, the resistance which may be overcome is lessened, and consequently the nail drives badly. It enters much better, however, if we place behind the plank some solid body which diminishes the recoil of the board.

29. We have considered hitherto forces which act for an appreciable time, but we now see that there may be forces which act for a very brief period, and yet in *this short time* produce or destroy a great momentum.

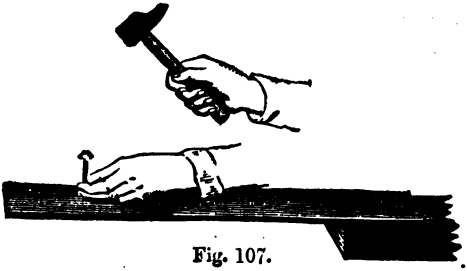


Fig. 107.

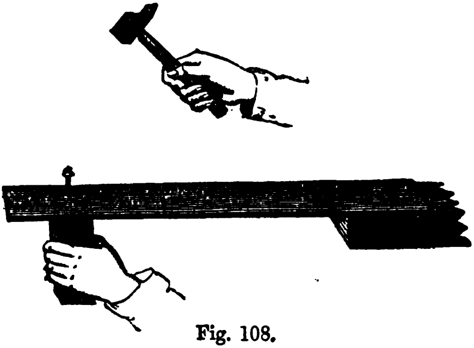


Fig. 108.

These are termed *impulsive* forces. *An impulsive force is one which produces a finite change of motion in an indefinitely short time.*

It is evident that such forces cannot be measured by the momentum generated in a unit of time; they are estimated by the whole momentum generated. Suppose, for example, that a blow given to a cricket ball of weight w drives it from the bat with a velocity v, then the momentum $v \times \frac{w}{g}$ is the measure of the force.

30. The first and second Laws of Motion explain the action of external forces on a body. By their means we are able to investigate the motion of a particle subjected to given forces. We have yet to examine the cases of motion which arise from the mutual action between or among two or more bodies. We shall be assisted in understanding the nature of this action by the following illustrations :—

(i.) When the block of the pile-driving engine (Fig. 106), after falling from A strikes B, it exerts a force on the pile which causes it for a very short interval of time to overcome the resistance offered by the soil. At the same time the block itself is acted on by a force upwards which brings it to rest, and these two forces are equal.

(ii.) If two balls moving in the same straight line are made to strike one another, the force of the first on the second is exactly equal to that of the second on the first, the force being measured by the momentum.

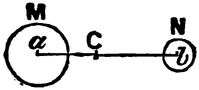


Fig. 109.

Let two balls M and N (Fig. 109) be approaching along the straight line *a b* with equal momentum, then it will be found that if they are not capable of rebounding, they will be reduced to rest after impact, the action on one being exactly equal and opposite to that on the other. Suppose, for example, that the balls weigh 6 lbs. and 8 lbs. respectively, and that their velocities are *4 ft. and 3 ft.* per second, so that the momentum of each is $24 \div g$, then they will both be brought to rest by the impact.

It is worthy of remark that in this case the C. G. of the two balls will not change its position during the motion; for let c be the point at which the balls meet, then $a\,c$ and $b\,c$ will be described in the same time, and therefore these lines must be proportional to the velocities of the balls. But since the momenta are equal, the velocities must be inversely proportional to the masses, and consequently c divides the line $a\,b$ into parts inversely proportional to the masses. Therefore c coincides with the C.G. of the two bodies, that is to say, the C.G. is the point at which the bodies will meet, and therefore has a fixed position.

(iii.) If a cannon free to move on a smooth plane be discharged, the cannon will recoil, and its momentum will be exactly equal to that of the shot, but in the opposite direction. Let w_1 be the weight of the shot, and v_1 its velocity on leaving the gun, and let w_2 be the weight of the gun, and v_2 its velocity of recoil, then

$$w_1\,v_1 = w_2\,v_2.$$

We need not write the g in the denominators, for they cancel.

(iv.) Suppose a shell to be moving on a horizontal plane with a velocity v, and suppose it to burst in the line of its motion into two parts, whose weights are w_1 and w_2. Let the velocity of the first part after explosion be less than the original velocity by v_1, then the velocity of the second part will be greater than the original velocity by v_2, such that

$$w_1 \cdot v_1 = w_2 \cdot v_2.$$

The momentum produced in one direction by the explosion will retard the motion of w_1, and that produced in the opposite direction will be equal to the former, and will increase the velocity of w_2. The sum of the momenta of the two parts will thus be the same before the explosion as after it. This we might have concluded also from the First Law of Motion, since by that law a body cannot of itself alter its momentum.

These are illustrations of a law which is known as Newton's Third Law of Motion. *To every action there is always an equal and contrary reaction.*

The momentum produced in any direction by the mutual action of two bodies being called *action*, the momentum produced at the same time in the opposite direction is termed *reaction.*

This principle was assumed in the case of statical forces in § 7, and also in § 15, p. 8, and § 25, p. 110, where the tension in a cord at rest or in motion was supposed to be the same at every point.

EXERCISES ON NEWTON'S THIRD LAW.

1. A body A of weight 10 lbs. strikes a body B at rest, and weighing 100 lbs., with a velocity of 100 ft. per second; find the velocity of B supposing A to be brought to rest by the impact.

2. Two bodies, whose weights are 6 lbs. and 10 lbs., and whose velocities are 50 ft. and 20 ft. per second respectively, approach, and after impact move on together; find the common velocity.

3. Three balls, weighing respectively 5 lbs., 7 lbs., and 8 lbs., lie in the same straight line. The first is made to impinge on the second with a velocity of 60 ft. per second without rebounding

The first and second together impinge in the same way on the third; find the final velocity.—*Ans.* 15 ft.

4. A gun weighing 5 tons is charged with a shot weighing 28 lbs.; if the gun be free to move, with what velocity will it recoil when the ball leaves it with a velocity of 100 ft. per second?—*Ans.* ·25 ft.

5. Two wooden balls weighing 12 oz. and 16 oz. are made to impinge on one another. One of the balls is furnished with a spike to prevent the rebound. If the velocity of the smaller be 12 ft. per second, what must that of the larger be that the motion may be destroyed by the impact?

6. A shell at rest bursts into two parts, the smaller of which is one-third of the whole; what will be the ratio of the initial velocities of the parts?

7. A shell moving with a velocity of 50 ft. per second bursts in the line of its motion into two parts, which weigh respectively 30 lbs. and 62 lbs. The velocity of the larger piece is increased in the direction of motion by 30 ft. per second; what is the velocity of the smaller?—*Ans.* 12 ft. in a direction opposite to the direction of motion immediately before the explosion.

8. Two bodies subjected to their mutual attraction, and to no other forces, start from rest. If their masses are as 5 to 2, and the acceleration of the larger is 40 ft. per second, find the acceleration of the smaller.

LONDON UNIVERSITY EXAMINATION PAPERS.

MATRICULATION.

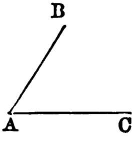
1. State the relation between the power and weight on a smooth inclined plane when the power acts parallel to the plane. What power sustains a man standing on a slope? (§ 88.)

2. Define the centre of gravity of a body. A uniform rod, three feet long, which weighs 3 lbs., has a 1 lb. weight hung from one end by a string one foot long: find the centre of gravity of the whole, and show where the rod must be supported in order that the whole may balance. (§ 54. The rod must be supported 13½ inches from one end, and when it is horizontal the C. G. will be 3 in. below this point.)

3. A cord without weight or friction passing round a single fixed pulley has a weight of 1,000 grains attached to one of its extremities, and one of 2,000 grains to the other: when left free, how far will this latter extremity have descended in two seconds?

$$\textit{Ans. } f = \frac{g}{3} \therefore s = \frac{2}{3}g.$$

4. Let A B and A C represent in direction and magnitude two equal forces acting at A, the angle between their directions being 50°. Draw a line which shall represent in direction and magnitude a force which will be in equilibrium with these forces; and state the angle which its direction will make with A B.



5. Explain why a man who has to carry a heavy box in one *hand must throw his* body on one side. (§ 66.)

If the force of gravity, instead of acting vertically, were to act horizontally from east to west, would this affect the position within a body of its centre of gravity? (§ 53.)

6. What is the ratio between the power and the weight in a screw which has ten threads to the inch, and is moved by a power acting perpendicularly to an arm at a distance of one foot from the centre?—*Ans.* $\frac{7}{5280}$.

7. How far will a body fall from rest in four seconds? With what velocity must a ball be thrown vertically upward, in order to return to the hand after four seconds?—*Ans.* 8 g, or 257·6 ft., and 2 g, or 64·4 ft.

8. A weight, W, is suspended from a movable pulley C, A and B being two fixed pulleys. When the system is at rest, what are the forces which keep the point C in equilibrium? and why is it necessary that the two weights W′ should be equal to one another? (§§ 18, 19.)

9. Two men, A and B, carry a weight of 200 lbs. on a pole between them. If the men be 5 feet apart, and the weight slung at a distance of 2 feet from A, what part of the weight will he bear, neglecting the weight of the pole?—*Ans.* 120 lbs.

10. Explain why a plumb-line will not remain at rest except in one position. (§ 54.)

11. An arrow is shot vertically upward with a velocity of 100 feet in a second when it leaves the bow. How long will it be before it reaches the ground again?—*Ans.* 200 ÷ g, or 6¼ secs.

12. If another arrow, half the weight of the former, be shot with the same *force* vertically upward, how long will it be before it reaches the ground again? Point out where you apply any of the laws of motion in answering this question.

(The moving force varies as the mass and as the velocity generated; hence, if the force remain the same, and the mass be doubled, the velocity will be diminished by one half, and therefore the time will be lessened in the same proportion.)

13. If a man has to raise a weight and has only one pulley at his disposal, show how he must apply it in order to obtain the utmost advantage.—*Ans.* Fig. 77, with parallel cords.

14. Two forces, not parallel, represented in magnitude and

direction by A P and B Q act at the extremity of a lever A B. Exhibit by means of a diagram the moments of these two forces round the fulcrum O, and state what condition will be satisfied when there is equilibrium (Fig. 60).

15. Show, from the property of the Centre of Gravity, that in a common balance it makes no difference in what part of the scale-pan the weight is put, whether in the centre or at the edge (§ 54).

16. A man who weighs 160 lbs. wishing to raise a rock, leans with his whole weight on one end of a horizontal crowbar 5 ft. long, which is propped at the distance of 4 in. from the end in contact with the rock; what force does he exert on the rock, and what pressure has the prop to sustain?—*Ans.* 2,240 lbs. 2,400 lbs.

17. A solid cube of wood rests by one of its sides upon an inclined plane in such a manner that the upper and lower edges of its base are horizontal. The inclination of the plane can be increased, and it is so rough that the cube will topple over before it will slide; find at what inclination this will take place (Fig. 49).—*Ans.* 45°.

18. How does it appear that the force of gravity acts on all substances alike?

19. A person inside clings to the roof of a railway carriage which then rushes horizontally over the edge of a precipice; what change, if any, in his motion will result if he lets go his hold? Give a reason for your reply.

20. A plummet, the string being held in the hand, is immersed in a current of water, and the string ultimately settles in a slanting position; explain by a diagram the nature and action of the forces which determine the position of the string.—*Ans.* The force of the current is horizontal, the weight vertical, and their resultant has the direction of the string.

21. How would you find the C. G. of a straight, but not uniform rod? (§ 53.)

22. A straight lever, 6 ft. long and heavier towards one end, is found to balance on a fulcrum 2 ft. from the heavier end; but *when placed* on a fulcrum at the middle, it requires a weight of

3 lbs. hung at the lighter end to keep it horizontal; what is the weight of the lever ?—*Ans.* 9 lbs.

23. Enunciate the Second Law of Motion. A ball is projected in a horizontal direction from a rifle placed 1,000 ft. above the level of the sea. Find the elevation of the ball two seconds after the discharge, neglecting the resistance of the air.—*Ans.* 1,000 — 2 g; or, 935·6 ft.

24. Exhibit, by means of a diagram of which the construction is to be explained, the method of finding the resultant of the forces represented by the lines in Figure 21, the forces all acting at the point M, and lying in the plane of the paper.

25. Find the Centre of Gravity of a system consisting of two spheres, 8 oz. and 24 oz. in weight, connected by a rigid rod without weight, the distance between the centres of the spheres being 1 foot.—*Ans.* 9 in. from the smaller weight.

26. Mention some of the most important applications of the screw.

27. State the Third Law of Motion, and give a numerical application of it.

Explain the "kick" of a gun.

28. A boat is moored in a stream by two ropes, fixed to posts, one on each bank, and inclined to the direction of the current at angles of 30° and 45°. Draw a figure from measuring which the proportion may be found between the strains on the two ropes.

29. A weight is to be raised by means of a rope passing round a horizontal cylinder 10 inches in diameter, turned by a winch with an arm 2½ ft. long. Find the greatest weight which a man could so raise without exerting a pressure of more than 50 lbs. on the handle of the winch.—*Ans.* A little less than 300 lbs.

30. Define the C. G. of a body. A triangular board is hung by a string attached to one corner. What point in the opposite side will be in a line with the string? (§ 56.)

31. When a body in motion is left wholly to itself, not being influenced by gravity or any other external force, what is the nature of the motion of translation of the body? Give some evidence from observation or experiment of the truth of your statement. (§§ 2, 3, 4.)

32. Explain by reference to a diagram why a stone only falls 16 ft. during the first second, while yet the force of gravity generates in that time the velocity of 32 ft. per second. (Fig. 114.)

33. Sketch a system of pulleys, one fixed and two movable, in which one end of the string passing round each pulley is attached to the weight, and find the relation of the power to the weight in equilibrium. (Fig. 81.)

34. A stone after falling for one second strikes a pane of glass, in breaking through which it loses half its velocity. How far will it fall in the next second?—*Ans.* g, or 32·2 ft.

35. It is required to substitute for a given vertical force, two forces, one horizontal, the other inclined at an angle of 45° to the vertical: determine the magnitudes of these two forces.—*Ans.* If P = the given force, the first = P, the second $\sqrt{2} \cdot P$.

36. How can the true weight of a body be determined by means of a balance with unequal arms? (§ 77.)

37. A body falling with a uniformly accelerated motion, passes through 10 ft. in the first two seconds after starting: how far will it be from the starting-point at the end of the third second? —*Ans.* 22½ ft.

38. Show how to find the resultant of two forces, A and B, which act upon a given point in directions that make an angle a with each other, the force A acting towards a given point, and the force B away from it. (§ 25.)

39. Six smooth vertical posts are fixed in the ground at equal intervals round the circumference of a circle, and a cord, without weight, is passed twice round them all in a horizontal plane, and pulled tight with a force of 100 lbs. Find the magnitude and direction of the resultant pressure on each post.—*Ans.* 200 lbs. towards the centre.

40. A straight lever 20 inches long weighs 15 oz. Where must the fulcrum be placed in order that the lever may be in equilibrium when a weight of 16 oz. is hung at one end, and a weight of 9 oz. at the other?—*Ans.* 8¼ in. from the 16 oz.

41. What is the pressure on the fulcrum when the lever is in *equilibrium?*—*Ans.* 40 lbs.

42. Draw a four-sided figure with unequal sides, and describe (1) a geometrical, and (2) an experimental method of finding its C. G. (§ 67.)

43. A piece of uniform paper in the form of a regular hexagon has one of the equilateral triangles obtained by joining the centre to two consecutive angular points cut out. Determine the position of the C. G. of the remainder of the paper. (§ 68.)

44. A heavy particle is dropped from a given point, and after it has fallen for one second another particle is dropped from the same point. What is the distance between the two particles when the first has been moving during five seconds?—*Ans.* 144 ft.

45. A balloon has been ascending vertically at a uniforn rate for 4·5 seconds, and a stone let fall from it reaches the ground in 7 seconds; find the velocity of the balloon, and the height from which the stone is let fall.—*Ans.* If S = the height, and V the velocity, then $S = V \times 4\cdot5$; also, $S = \frac{1}{2} g\, 7^2$; hence, $V = 5\cdot\dot{4} \times g$, and $S = 24\cdot5\, g$.

46. In a system of one fixed and four movable pulleys, in which one end of each string is fixed to a beam, find the relation between the power and the weight (neglecting the weight of the pulleys) when one of the strings is nailed to the pulley round which it passes.

What is the force exerted on the beam to which the strings are attached?—*Ans.* When the string is nailed to one pulley, that pulley becomes nothing more than a link, and the system, when used for *raising* the weight, is reduced to three movable pulleys, hence $P = \frac{1}{8} W$.

47. Two uniform cylinders of the same material, one of them 8 in. long and 2 in. in diameter, the other 6 in. long and 3 in. in diameter, are joined together, end to end, so that their axes are in the same straight line; find the centre of gravity of the combination.—*Ans.* $5\frac{26}{43}$ in. from the base of the shorter.

48. If two forces acting on a point are represented in magnitude and direction by two sides of a triangle, under what circumstances will the third side correctly represent their resultant? Forces of 20 and 10 act along the sides A B and B C respectively of an equilateral triangle; find the magnitude of their resultant. *(§ 25.)*

1870.—JANUARY.

1. Show how to find the resultant of three given forces acting on a point; and prove that, to produce equilibrium, their directions must be in the same plane.

2. Find the ratio of the power to the weight for equilibrium on a bent lever of the first kind, when the forces act at right angles to the arms. Supposing the arms make an angle of 120° with each other, and have the relative lengths of 1 and 5; find the magnitude and point of application of the resultant of the power and weight when the lever is in equilibrium.

3. Two heavy particles, weighing respectively 3 and 5 ounces, are attached to the ends of a straight rod 8 inches long weighing 2 ounces; find the centre of gravity of the system.—*Ans.* 4·8 inches from the smaller weight.

4. A body of given mass is acted upon by a constant force; find the space described in a given time.

If a particle move from rest through 40·5 feet in $4\frac{1}{2}$ seconds under the action of a constant force, find the acceleration.—*Ans.* 4 ft.

1870.—JUNE.

1. What is meant by the moment of a force about a given point? How is its magnitude determined?

If several forces, which do not balance, act in the same plane upon different points of a solid body one point of which is fixed, what condition must be fulfilled in order that the body may be in equilibrium? Show that this condition cannot be fulfilled unless the fixed point is in the plane of the forces.

2. What condition must be fulfilled in order that a body may be in equilibrium upon a hard, smooth, plane surface?

A weight rests upon a smooth horizontal plane, and is acted on by a force, equal to the weight of 6 lbs., in a direction inclined obliquely downwards at an angle of 30° to the horizon. Find the magnitude of the horizontal force required to produce motion.

3. Find the relation between the power and the weight in the Third System of Pulleys (in which each string is attached to the weight), when the weights of the pulleys are disregarded. Also, show what the relation would be if the weights of the pulleys were taken into account.

4. What is meant by the "acceleration" due to a force; and upon what does its magnitude depend? If the velocity of a body increase from 12 to 13 feet per second while it moves over a distance of 5 feet, what is the acceleration? Indicate the course of the reasoning upon which your calculation is based.—*Ans.* 2·5 ft.

1871.

1. Show that as the angle between two forces is increased, their resultant is diminished.

Two forces of the magnitudes 5 and 11 act at angles of 60°, 90°, and 120° respectively: compare their resultants in the three cases.—*Ans.* $\sqrt{201} : \sqrt{146} : \sqrt{91}$.

2. A weight of 56 lbs. is attached to a straight lever without weight, at a distance of 3 inches from the fulcrum, and is balanced in one case by a power of 6 lbs., and in another case by a power of 16 lbs.: find in each case the pressure on the fulcrum, and also the distance between the points of application of the power and weight, when they are applied—

(a) on the same side as the fulcrum,

(b) on the opposite sides of the fulcrum.

Show how the answer to each part of the question would be affected if the lever weighed 9 lbs., and its centre of gravity were at the fulcrum.

3. Four heavy particles, of the relative weights 2, 3, 4 and 5, are placed at the corners A, B, C and D respectively, of a horizontal square board: find the common centre of gravity. Would its position change if the board were inclined?

4. If a heavy body is thrown vertically up to a given height, and then falls back to the earth, show that, neglecting the resistance of the air, it passes each point of its path with the same velocity when rising and when falling.

A ball is allowed to fall to the ground from a certain height, and at the same instant another ball is thrown upwards with just sufficient velocity to carry it to the height from which the first one falls: show when and where the two balls will pass each other.

5. A mass originally at rest is acted on by a force which in 1-368th of a second gives to it a velocity of $5\frac{1}{4}$ inches per second: show what proportion the force bears to the weight of the mass.

—*Ans.* Acceleration = 161 ft. $\therefore \frac{P}{W} = \frac{161}{g} = 5.$

HYDROSTATICS.

I.—INTRODUCTION.

1. All matter with which we are acquainted presents itself in one of two forms, either *solid*, as iron, wood, stone, or *fluid*, as air, water, oil.

When a knife is used to separate into parts a block of wood, however sharp the knife may be, it will not enter the block unless a certain pressure be applied to it; but if a very thin plate of metal be placed in air or water, very little resistance will be experienced to its motion in the direction of its plane. This difference distinguishes the two classes of bodies. The particles of a solid body cohere, so that a force is necessary to separate them; the particles of a fluid may be separated by the slightest possible force. Hence a fluid differs from a solid by the absence of cohesion and friction. Although there is no fluid in nature between the particles of which there is an absolute absence of friction, nevertheless those substances which we shall consider as fluid fulfil this condition so nearly, as to make the conclusions founded on it practically correct. Hence we shall define a *fluid* as *a collection of material particles which move freely amongst themselves, and consequently can be displaced by the slightest force.*

2. There are two classes of fluids—gases and liquids. The particles of the former not only yield to the smallest pressure, but have a repulsive force among themselves. There is no such repulsive force between the particles of a liquid. A cubic foot of any gas may readily be compressed into half a foot, double the pressure will reduce it to a quarter of a foot, and when the pressure is removed the gas returns to its original bulk; but no ordinary pressure produces any sensible compression on water or any other liquid. Hence gases have been denominated *elastic* fluids, and liquids *non-elastic.*

3. The perfect mobility of the particles of a fluid leads to special conditions of equilibrium. The science which treats of these conditions is termed Hydrostatics.

4. We have said that liquids cannot be compressed by any ordinary force; and this is so far true, that both in the theory of hydrostatics and in practice, they are assumed to be perfectly incompressible. But under great pressure liquids are slightly compressible. By sinking a vessel filled with fluid in the ocean to the depth of 2000 metres, where every square centimetre supports a weight of 200 kilogrammes, and every square inch nearly 3000 lbs., it is found that the volume of water is diminished one-twentieth.

5. If the pressure of the atmosphere be removed from a given volume of water, its bulk increases by ·00005 of this volume. The increase in the case of mercury would be nearly ·000005 of its original bulk.

II.—THE TRANSMISSION OF PRESSURE.

6. In mechanics we have considered the action of forces on rigid or solid bodies. We have found, for example, that when two parallel forces are applied at the ends of a lever, there will be equilibrium if one particular point in the lever be fixed. This is a property of solid bodies. When, however, the forces act on fluids, the result is different. Each point in the fluid must be directly sustained.

Let us imagine a straight tube full of water, with a piston fitting one end. If a pressure of 1 lb. be exerted on the piston, the water will escape from the other end of the tube; to prevent this, a pressure of 1 lb. must be exerted in the opposite direction by means of an air-tight piston like the first. There is here nothing new: if instead of pistons and fluid we had a rigid rod of wood or metal, the same relation between the forces would produce equilibrium. But let us bend the tube in any way we please (Fig. 1), and the two equal pistons, A and B, placed at the two ends, will be in equilibrium if acted on perpendicularly to their surfaces by equal forces. It is not necessary that the forces should here be directly opposite. The pressure exerted by the piston A is transmitted through the windings of the tube, and acts perpendicularly to the surface of the piston B, even when the direction is entirely different from

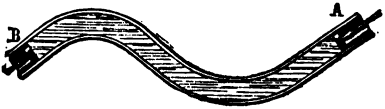


Fig. 1.

that of A. It is to this pressure at B that the force applied must be equal and opposite.

7. It must be remembered that we are speaking of the equilibrium of the fluid in the tube, and not of the tube itself. We suppose the tube to be fixed firmly on its support. The equilibrium of the solid tube and the equilibrium of the fluid are distinct things, and must be considered separately.

8. Take a closed vessel of water, and make in it an aperture B (Fig. 2), into which a piston with a surface of one square inch may be fitted, and exert a pressure of 1 lb. on this piston. We may imagine that a tube passes perpendicularly from the aperture B, traversing the mass of the liquid, and meeting the surface of the vessel again perpendicularly at a point A. Since there is equilibrium the fluid surrounding the portion A B must exert everywhere a pressure upon this portion equal to that which would be exerted by the surface marked off if it were that of a rigid tube. Hence, the pressure at B will be transmitted to A, and, in order that it may be in equilibrium, a force of 1 lb. must be applied *externally.*

As we may repeat the same hypothesis for each point in the surface of the vessel, we conclude that the pressure exerted on B is transmitted by the liquid equally in every direction, so that every square inch of surface is acted on by a normal pressure of 1 lb.

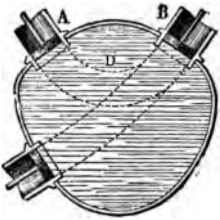


Fig. 2.

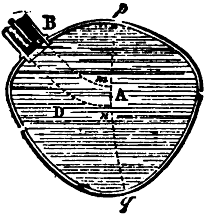


Fig. 3.

9. Imagine a surface *p q* (Fig. 3), within the liquid, and suppose a tube constructed from B to a point A in the surface, so as to cover at that point an area, *m n*, equal to one square inch; the surface *m n* will sustain a pressure of 1 lb. Moreover, since the mass of the liquid itself is in equilibrium, it follows that *m n* must also sustain a pressure of 1 lb. in the opposite direction. This important principle of fluids may be enunciated thus: *A pressure exerted at any point whatever in a fluid in equilibrium is transmitted undiminished to every point in the fluid.*

10. If there are two pistons, B and C, each equal to A, then when a pressure of 1 lb. is applied at A, a pressure of 1 lb. must also be applied at B and C. Suppose the pistons united so that they present the same area to the fluid, they will sustain the same

pressure—that is to say, twice the pressure on A. A piston having three times the area of A will sustain thrice the pressure, and generally the pressures sustained are proportional to the areas.

11. Suppose, for example, that the pistons are fitted to two tubes connected by a space filled with water (Fig. 4). Let the area of the smaller piston be one square unit, and that of the larger ten square units. When the smaller piston is pressed in with a force of 1 lb., it will be necessary to exert a pressure on the larger of 10 lbs., in order to prevent it from being moved. If a and b be the areas, the pressure on the first is to the pressure on the second as $a : b$. If the radius diameter or circumference of one piston be n times the corresponding line of the other, the area of the first will be n^2 times the second, and the pressure on the first will therefore be n^2 times that on the second.*

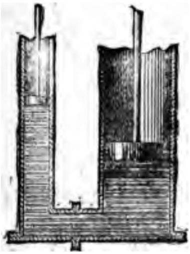


Fig. 4.

Example.—If the circumference of the smaller be 2 in., and that of the larger 15 in., the pressure p on the former is to the pressure P on the latter as $2^2 : 15^2$. Let the pressure on the former be 1 lb., then that on the latter is $15^2 \div 2^2$, or 56·25 lbs.

12. The principle of virtual velocities applies to this as to every other mechanical power. Suppose,

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* It must be remembered that the *ratio* of the areas is what is here wanted, the area of a circle is the square of the radius *multiplied by π* (or approximately, by $\frac{22}{7}$).

for example, the small piston to descend n in., the amount of water driven out of the smaller cylinder into the larger will be $a \times n$ cubic in. If the larger piston ascend m in., the amount of water forced into it will be $b \times m$; hence

$$\begin{array}{lrcl} & a \times n & = & b \times m \\ \text{but} & b \times p & = & a \times P \\ \text{hence} & p \times n & = & P \times m \end{array}$$

The principle explained above is applied in the Bramah or Hydraulic Press.

The Bramah Press.

13. *Description of the Press.*—The whole instrument is represented in Fig. 5, and a section of it in

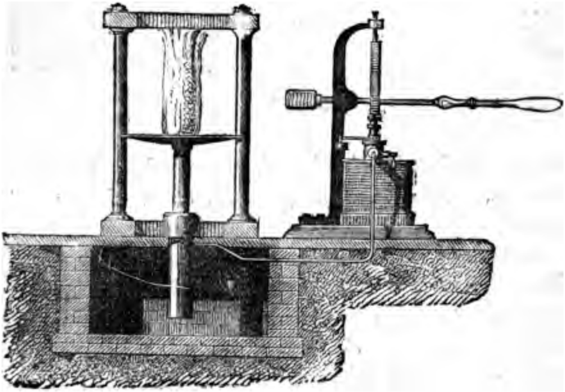

Fig. 5.

Fig. 6. a and b are two pistons working through air-tight collars in strong cylinders filled with water, and connected by a tube c c. The small piston a is worked by a lever represented on the right of the upper figure, and the larger piston *b* is attached to a movable platform, on which the substance to be pressed is placed. At *h* and *r* are valves opening upwards. The tube *d* passes into a cistern of water.

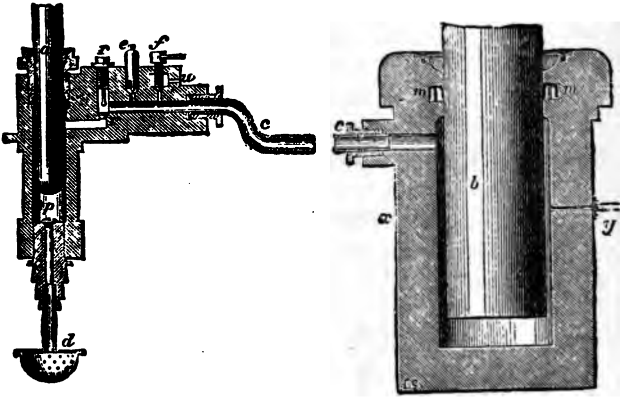

Fig. 6.

Action of the Press.—Suppose the cylinder filled with water, and *a* in its lowest position; on raising *a*, the atmospheric pressure forces water from the reservoir through *d*, and when *a* is afterwards forced down, the valve *p* closes, the valve *r* is opened, a portion of the water is driven along *c*, and the cylinder *b* is made to ascend. By repeating this process any required compression of the substance between the platforms may be produced. At *f* there is a plug, which can be unscrewed when the compression is completed.

14. The press is named from Mr. Bramah, who invented the simple and beautiful contrivance represented at *m m* in Fig. 6, for preventing the escape of the water from the cylinder. The ring or collar *m* is lined with leather, which is allowed to hang down like a flap in the cylinder. When water is forced upwards between the piston and the cylinder, it fills the fold of leather, and causes the edge to embrace the piston, so that the liquid cannot force its way higher than *m*. The greater the pressure, the more tightly will the flap close round the piston.

The relation between the power exerted on the press and the force at the end of the lever.

15. Let the diameter of the piston in the pump be 1·5 centimetres, and the diameter of the piston of the press 20 centimetres. Let the force be 18 kilogs., applied at 36 centimetres from the fulcrum, and let the piston of the pump be 6 centimetres from the fulcrum.

Force at a distance of 36 cents. from fulcrum $= 18$
,, ,, 6 ,, ,, $= 108$
$\therefore$ Pressure on area $(1{\cdot}5)^2 = 108$
,, $1 = \frac{108}{2{\cdot}25} = 48$
,, $20^2 = 48 \times 400 = 19{,}200$ kilogs.

16. In the same way we may find the relation for any dimensions.

Let *m* and *n* be the arms of the lever, F the force acting on the lever, P the pressure exerted by the piston of the pump, a and b the areas of the pistons. Then, by the principle of the lever,

the pressure on the smaller piston $= F.\frac{m}{n}$

the pressure on one unit of area $= F.\frac{m}{na}$

pressure on b units of area $= F\frac{mb}{na}$

If R and r be the radii, $\frac{b}{a} = \frac{R^2}{r^2}$

Hence the pressure on $b = F\frac{m}{n}\frac{R^2}{r^2}$.

Instead of a pressure of 1 lb. on the piston (Fig. 4), a pound of water in the tube above would produce the same effect. If the vertical tube holding this water be made twice as long, but of half the sectional area, the pressure on a unit of area will be doubled, while the same amount of fluid only will be used. By continually repeating this process, the small quantity of fluid may be made to support a very great weight. This principle, stated as follows, is sometimes termed the Hydrostatic Paradox:—*Any quantity of fluid, however small, may be made to support any weight, however large.*

EXERCISES ON THE PROPERTIES OF FLUIDS.

1. Why does a liquid readily change its form when its position is changed?

2. What properties of liquids are not possessed by solids?

3. If a solid be placed inside a sphere, and a pressure be exerted upon the solid by a piston, how will the pressure be transmitted?

4. If the sphere be filled with a liquid, and pressure be exerted by means of the piston, how will the pressure act on the sphere?

5. If a vessel full of liquid be provided with two pistons, one of which is 20 square centimetres in area, and the other 100 square centimetres, and a pressure of 3 kilogs. be applied to the former, what pressure will be exerted on the latter?

$$\text{Pressure on } 20 \text{ square c.} = 3 \text{ kilogs.}$$

$$\text{,, on } 1 \text{ square c.} = \frac{3}{20}$$

$$\text{,, on } 100 \text{ square c.} = \frac{3 \times 100}{20} = 15.$$

6. If one piston be 10 centimetres in circumference, and the other 80, what must be the pressure applied to the smaller that the larger may support 1000 kilogs.?

$$\text{Pressure on area of } 80^2 = 1000$$

$$\text{,, ,, } 1 = \frac{1000}{6400}$$

$$\text{,, ,, } 10^2 = \frac{1000 \times 100}{6400} = 15\tfrac{5}{8}$$

7. If the area of the smaller piston be 12, and that of the larger 300, what pressure will the larger sustain when 5 kilogs. is applied to the smaller?

8. The diameter of the smaller piston being 3, and that of the larger 18, what pressure on the smaller will produce a pressure of 900 lbs. on the larger?—*Ans.* 25 lbs.

9. The circumferences of the pistons are as 4 to 55, and the pressure on the smaller 400 grammes, find that on the larger.—*Ans.* 75,625 grammes.

10. In descending 20 centimetres, the smaller piston raises a weight of 800 lbs. through a height of ·5 centimetre. What pressure is applied?—*Ans.* 20 lbs.

11. Two vertical tubes A and B are connected by a horizontal tube; A is short, and is provided with a piston; B is long, and has no piston. The diameter of A is 50 times that of B, what height of water in B will support 1000 kilogs. in the piston in A (a depth of a centimeter produces a pressure of one gramme on a square centimetre)?—*Ans.* 4 metres.

12. The pressure on a plane area in the form of a square, the side of which is a yard, is known to be uniform, and to be equal

to 2700 lbs.; find the pressure at a point when the unit of length is a sq. in.—*Ans.* 2$\frac{1}{12}$ lbs.

13. The pressure at any point in a rectangular surface 1·5 metres long and 1·2 metres broad, is 100 grammes, a centimetre being the unit of length. Find the pressure on the whole surface. —*Ans.* 1800 kilogs.

14. If the diameter of the pump of a Bramah press be 3 centimetres, and that of the press 120 centimetres, what force will 1 kilog. applied to the pump produce on the press?—*Ans.* 1600 kilogs.

15. If a power of 100 kilogs. be applied to a lever at 30 centimetres from the fulcrum, the piston of the pump being attached 6 centimetres from the fulcrum, what will be the force acting on the press in the last question?—*Ans.* 800,000 kilogs.

III.—PRESSURE OF A FLUID ARISING FROM ITS WEIGHT.

17. Liquids have weight like solids; but the particles of a liquid glide over each other without friction. Hence, when a liquid is contained in a vessel, it is always found that the surface is in a horizontal plane. When the fluid is at rest, we see that this must be the case. If part of the fluid surface were inclined (Fig. 7), we should have particles acted on only by their weight at rest, on an inclined plane *free from* friction, which cannot be.

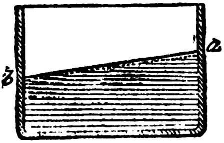

Fig. 7.

18. When the liquid is contained in several vessels communicating with each other, it is still found that the surfaces in the different vessels all lie in the same horizontal plane. Thus, if tubes of various shapes be connected as in the figure (Fig. 8), the water rises in all to the same level. This fact is frequently expressed by saying that liquids maintain their level. If a small vertical tube *t* communicating with the vessels terminate below the level of the surface, and be pierced with a small hole at the top, a jet of water rises from the orifice to the level of the surface. The force with which the water leaves the aperture is proportional to the depth of the aperture below the surface.

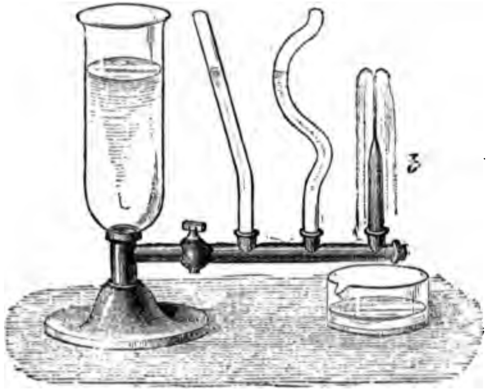

Fig. 8.

19. This principle is applied in many ways. Water is distributed over a city by subterranean

pipes issuing from a large reservoir situated on an adjacent elevation. These pipes are terminated by stop-cocks, which are below the surface of the reservoir, so that when they are turned on the water escapes.

20. The law according to which the pressure produced by the weight of a fluid is transmitted in every direction frequently makes the effect of this weight very different from that of a solid. Suppose, for example, that a cylinder of ice is placed in a vessel of the same shape, the weight of the ice will exert a pressure on the base, but none on the sides of the vessel. The sides may be removed, while the mass within retains the same position. When, however the ice is melted, there is not only a pressure on the base equal to the weight, but also a pressure on the sides. If a hole be made in the vessel, the liquid will flow out, and the force with which it escapes will be proportional to the depth of the hole.

This variation in the pressure may be illustrated by the following experiment.

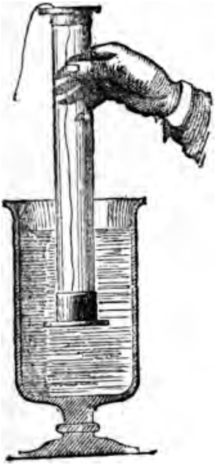

Fig. 9.

21. Take a hollow cylinder or tube of glass, one end of which may be closed by a disc of metal supported *by a thread* (*Fig.* 9). Apply the disc to the tube,

and plunge the closed end in a vessel of water. At a certain depth, depending on the weight of the disc, the thread may be dropped, and the disc supported by the fluid pressure only. If the cylinder be moved horizontally, no change takes place. This result illustrates the fact that the pressure at points in the same horizontal plane are equal. If a heavier disc be taken, it must be plunged to a greater depth before it will be supported, and it will be found that the weight supported is proportional to the depth. Suppose, now, we take a very thin disc, make the cylinder descend to a certain depth, then pour water gently into the tube, we shall find that when the water in the tube reaches the level of that outside, the disc will be detached; being equally pressed on both sides, it falls by virtue of its weight. Hence the upward pressure on the disc is the weight of the liquid which would fill the cylinder to the level of the surrounding fluid.

22. Suppose the area of the disc and cylinder to be 40 square centimetres. The weight of a cubic centimetre is 1 gramme, and, with a base of 40 square centimetres, and height 1 centimetre, we get 40 cubic centimetres. Hence, at a depth of 1 centimetre, a disc weighing 40 grammes will be supported. At a depth of 2 centimetres the pressure of the fluid will be twice 40 grammes, and so on. Thus, for every square centimetre of area in the section of the cylinder there will be a pressure of as many grammes as there are centimetres in the depth of the disc below the surface.

23. The principle that the pressure depends on

the depth, and not on the quantity of fluid, may be further illustrated by the following series of experiments. Take a balance having a circular disc of metal attached by a fine wire (Fig. 10) to one end of the beam, just heavy enough to balance the scale at the other. Take a cylindrical vessel open at both

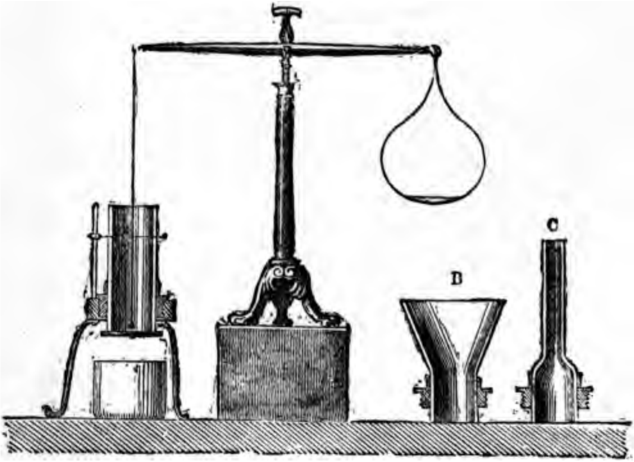

Fig. 10.

ends, and place it with axis vertical on a tripod, so that the wire supporting the disc passes along the axis of the cylinder. Let the disc and cylinder be properly ground, so that when pressed together they form a vessel which will hold water. Place a certain weight, a pound, for example, in the scale; the disc will now be pressed against the cylinder, and, if water be *poured gently* into the cylinder, the disc will remain in contact until the water rises to a certain height. If more water be added, the pressure will overcome the weight in the scale, and will force open the bottom. On a rod attached to the stand mark the height of the water when the highest level is reached. It will be found that exactly one pound of water will be supported. Pour the water out of the cylinder into the scale, taking away the 1 lb. weight, replace the cylinder, and fill it again gently up to the height *h*; the water in the scale will be just sufficient to keep the disc in contact with the cylinder. We see, therefore, that the pressure on the disc is equal to the weight of the column of water above it.

Repeat the experiment with vessels B and C (Fig. 10) of different shapes, having the area of the base the same. Water must be poured to the same height as before, in order that the pressure on the disc may be 1 lb. We shall find, however, that with vessel B there is more than 1 lb. of water, and with vessel C less than 1 lb. Thus the pressure on a plane does not depend on the amount of water used; it depends on the size of the plane and its depth below the surface. In all the cases the pressure is equal to the weight of a column of fluid having the plane for base and the depth of the plane for height. Only in the case of the cylinder is this the same as the weight of the water used. Thus we see that the pressure on the horizontal base of any vessel is the weight of the fluid which a vessel having the same base and vertical sides would contain, when filled to the same level.

Example.—What is the *pressure* on the base of a cone which has the vertex upwards and is filled with water to the depth of 18 centimetres, the radius of the base being 14 centimetres?

The area of the base is $14^2 \times \pi$ or $\frac{22}{7}$ of $14^2 =$ 616 square centimetres.

The volume of fluid which would be contained in a cylinder on the same base, and having the height of the water in the cone $= 616 \times 18 = 11{,}088$ cubic centimetres. Hence the pressure is 11,088 grammes.

The Pressure on the Sides of a Vessel.

24. The pressure on the sides of a vessel will not be uniform, but will be less near the top and greater near the bottom. Let us consider a vessel, the sides of which are rectangles. The pressure increases with the depth, and the mean pressure is that at half the depth. Hence the pressure on the rectangle would be equal to the weight of water above the rectangle, placed horizontally in a liquid at a depth equal to the depth of its middle point.

Example.—The rectangular side of a vessel is 50 centimetres long, and the water in contact with it is 20 centimetres deep. Find the pressure on the side.

The area equals $50 \times 20 = 1000$ square centimetres. The depth of the middle point is 10 C. Hence the pressure is equal to the weight of 1000×10, or 10,000 cubic centimetres of water; that is, 10,000 grammes.

25. If the surface be not a rectangle, the mean pressure will be that at the centre of gravity of the *surface.* Hence, if we suppose the side spread out

horizontally, and a column of the fluid resting on it to a height equal to the depth of the centre of gravity of the side, the weight of this column will be equal to the pressure on the side.

Example.—Find the pressure on a triangle whose base is 12, height 10, and centre of gravity at a depth of 30 centimetres.

$$\text{Area} = \frac{1}{2} \times 12 \times 10 = 60.$$

Pressure at depth of 1 cent. = 60 grammes.
" " 30 " = 1800 "

26. When embankments are made to streams or reservoirs they must increase in breadth and strength with the depth. The breadth of water makes no difference in the pressure when the water stands at the same level on both sides. The pressure on a dock-gate is the same on the inside, where the water is only a few yards across, as on the outside which faces the ocean.

27. The line of action of the resultant pressure on a surface will intersect the surface in a certain point, termed the centre of pressure. Suppose it is required to place a prop against the vertical side of a cistern, so as to give it additional support, at what height must the prop be applied? If the pressure on the rectangle forming the side were uniform, then the required point would be the centre, but the pressure varies as the depth, and is, therefore, greater on the lower half than on the upper, so that the prop must be applied at some point below the centre. To find the exact position of this point is to find the centre of pressure of the rectangle.

28. The centre of pressure P of a rectangle having its upper side in the surface of a fluid, is found to be in the line joining the middle points of its horizontal sides, at one-third from the base. Thus, if one side of the rectangle be under the action of the fluid pressure, and the other side be acted on only by a single force equal to the fluid pressure at the point P, the rectangle will remain at rest.

This is an important principle which has to be borne in mind in placing the hinges and supports ot such surfaces as lock-gates.

Note.—In the following Exercises it is frequently necessary to connect units of volume with units of weight.

In the metric system, the principal unit of length is the *metre*. The unit, however, which is most frequently used in considering fluid pressures is the hundredth part of the metre, or the *centimetre*. The exact weight of a cubic centimetre of pure water at a certain definite temperature (4°) is one *gramme*. When we speak of the weight of a certain bulk of water, we shall always suppose the water to be pure, and to have a temperature of 4°. A kilogramme is 1000 grammes.

When English units are used, we cannot express *exactly* the relation between the volume of a cubic unit of water and its weight.

A cubic foot of water weighs *nearly* 1000 ounces.

A cubic inch ,, ,, $\frac{1000}{1728}$ ounces.

EXERCISES.

1. Find the pressure at the depth of 30 metres in a lake.—*Ans.* 3,000 grammes on a square centimetre.

2. Find the pressure on a square inch at a depth of 600 feet in a lake, the weight of a cubic foot being 1000 ozs.—*Ans.* $4166\frac{2}{3}$ ozs.

3. In the experiment illustrated in Fig. 9. if the section of the cylinder be 36 square centimetres in area, and the weight of the disc 1080 grammes, to what depth must it be sunk before the thread may be dropped?—*Ans.* 30.

4. Find the pressure on a surface, 18 units by 15, at a depth of 20, the weight of a cubic unit of water being 10 lbs.—*Ans.* 54,000 lbs.

5. Two cubical vessels, the edges of which are as two to one, are filled with water, compare the pressure on the bases.—*Ans.* 8 to 1.

6. A cubic inch of mercury weighs 8 ozs., what will be the pressure on the bottom of a vessel, the area of which is a square foot, the depth below the surface being 5 inches?—*Ans.* 360 lbs.

7. A plate of metal is held at the end of a hollow glass cylinder 6 centimetres in diameter, while the cylinder is immersed vertically. It is found that the plate is just supported at a depth of 35 centimetres; find the weight of the plate in grammes.—*Ans.* 990 grammes.

8. If two vessels have equal bases and the same depth, but one widens upwards to the mouth, and the other diminishes upwards to the mouth, on which is the pressure on the base the greater?

9. What will be the pressure on a rectangle 11 by 60 inches, immersed so that a diagonal is vertical in a fluid a cubic inch of which weighs $1\frac{1}{2}$ oz.?—*Ans.* 30,195 ozs.

10. What will be the pressure on a vertical triangle immersed in water with the base in the surface, the base being 50 centimetres, and height 30?—*Ans.* 7500 grammes.

11. The upper surface of a vessel filled with water is a square whose side is 8 metres, and a pipe communicating with the interior is filled with water to a height of 2 metres above the surface; find the weight which must be placed on the lid of the vessel to prevent the water from escaping.—*Ans.* 128,000 kilogs.

12. A house is supplied with water from a reservoir, 241 feet above the ground floor, by means of a pipe laid underground, what will be the pressure per square inch on a tap 25 feet above the

ground floor, supposing a cubic foot of water to weigh 1000 ozs.?—*Ans.* 1500 ozs.

13. A vessel is filled with water, how will the pressure on the base be affected if a piece of metal be dipped into the water?

14. If the vessel be not full, how will the pressure on the base be affected by dipping a piece of iron into the water?

IV.—THE PRESSURE ON A SOLID IMMERSED IN A LIQUID.

29. When bodies are placed in a fluid, it is a matter of common observation that they will sink or float, according as they are, bulk for bulk, heavier or lighter than the fluid. Even when a body sinks, it appears to be of less weight in water than in air.

For example, it is easy for a person to raise a mass of stone under water which he is unable to lift above the surface.

The pressures exerted by the fluid on the surface of the body have a resultant which acts in a direction opposite to that of the weight of the body. If the weight of the body exceed the fluid pressure, the body sinks; if the pressure be greater than the weight, the body ascends; when the weight and the pressure are equal, the body rests. These different conditions may be made to succeed each other at pleasure, with the toy represented in Fig. 11. A *hollow globe of* glass, containing air and water, floats on the surface of the water in a glass cylinder, over the top of which is stretched an elastic cover. In the lower part of the globe there is a very small hole, by which water can enter or escape. If the hand be placed on the elastic top, the air in the cylinder is compressed, and its pressure on the surface of the water is increased. This augmented pressure is transmitted through the water to the air in the globe, which is condensed. The consequence is that a small quantity of water enters the globe, increases its weight, and causes it to sink. When the pressure is removed, the bulb rises again. The pressure may be so regulated that the bulb shall rest suspended in any part of the water.

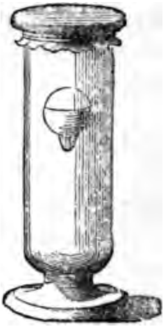

Fig. 11.

30. The law according to which a fluid exerts a pressure on a body immersed is as follows:—

When a solid is immersed in a liquid, the resultant pressure on it is the weight of its own bulk of liquid.

Let A be a solid immersed in water (Fig. 12). Imagine the space A filled with water, and then suppose the mass A to become solid; it will remain suspended like any other part of the fluid. Since its weight acts downwards, it is evident that there is an equal force acting upwards, counteracting the weight. This force is the resultant of all the fluid pressures, acting in all directions, upward,

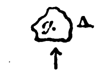

Fig. 12.

downward, and laterally. Some of these pressures neutralize others; but there must remain a balance of upward pressure, since the under surface is at a greater depth than the upper, and it is this resultant pressure which counteracts the weight of the imagined isolated mass. Therefore the upward pressure on A is equal to the weight of A. Since two forces in equilibrium must act in the same straight line, the resultant must pass through the centre of gravity, g, of the mass.

If we imagine the ice to have become stone without altering its bulk or its surface, the pressure acting upon it will still be the same, namely, the weight of the same bulk of water as would exactly fill the space occupied by the stone. Thus the weight of the body is diminished by the weight of the water displaced.

V.—SPECIFIC GRAVITY.

31. Equal bulks of different substances have different weights. If equal volumes be taken, lead is heavier than iron, iron is heavier than water.

It is a problem of great practical importance to determine the relation between the weights of different substances. For this purpose all substances are compared with water, and the relative weight of any substance is expressed by the number of times *the weight of* the given body contains the weight of an equal bulk of water. This number of times is called the specific gravity of the body.

32. *The measure of the specific gravity of a body is the ratio of the weight of any volume of the body to the weight of the same volume of a certain standard substance.*

Water by its universal distribution and the ease with which any required volume is obtained and purified, is best adapted for the standard substance.

Hence if s be the specific gravity of a body, the weight of the body over the weight of an equal bulk of water $= s$.

Let V be the volume of the body, and let w be the weight of a unit of water, then the weight of the same bulk of water $= wV$, thus:—

$$\frac{W}{wV} = s \text{ or } W = wVs$$

If the unit of volume be a cubic centimetre of distilled water at 4° $w =$ one gramme; if the unit of volume be a cubic foot $w = 1000$ ozs. nearly.

Thus, if we know the specific gravity of a body we can find the weight of any volume.

Example 1.—What is the weight of 48 cubic inches of copper the specific gravity of which is 8·82.

Weight of 48 cubic inches of water $= \frac{48 \times 1000}{1728}$

Hence, weight of 48 cubic inches of substance, whose specific gravity is 8·82

$$= \frac{48 \times 1000 \times 8{\cdot}82}{1728} = 245 \text{ ozs.}$$

Example 2.—What is the weight of 250 cubic centimetres of mercury, the relative weight being 13·596?

1 cubic centimetre of water weighs 1 gramme,
1 cubic centimetre of a substance of specific gravity 13·596 weighs . . . 13·596 grammes,
250 cubic centimetres weigh $250 \times 13\cdot596$ grammes,
Or 3399 grammes.

Example 3.—What volume of lead of specific gravity 11·4 will weigh 190 grammes?

11·4 grammes of lead have the same bulk as 1 gramme of water.

Hence 11·4 grammes of lead measure 1 cu. cent.
1 gramme „ „ $\frac{1}{11\cdot4}$ „
190 grammes „ „ $\frac{190}{11\cdot4}$.
Or 16⅔ cubic centimetres.

Example 4.—A piece of marble of specific gravity 2·7 weighs half a ton. What is its size?

Half a ton=17,920 ozs.
Volume of 1000 ozs. of water = 1 cu. ft.
„ 1 oz. „ = ·001 „
„ half a ton „ =17920 × ·001 cu. ft.
=17·92 cu. ft.

Volume of the same weight of a substance whose sp. gr. is 2·7 $\Bigg\} = \frac{17\cdot92}{2\cdot7} = 6\cdot06$ cu. ft.

Example 5.—Find the specific gravity of a *mixture of* 1 cubic centimetre of a fluid of specific gravity 1·3, and two cubic centimetres of specific gravity 2·2.

Weight of 1st, 1·3 grammes.

Weight of 2nd, twice 2·2 or 4·4

Weight of mixture, 5·7 grammes.

Volume of mixture=3 cubic centimetres, and weight of equal bulk of water=3 grammes.

Hence specific gravity of mixture$=\frac{5 \cdot 7}{3}=1 \cdot 9$.

Example 6.—To find the specific gravity of a mixture of given volumes of any number of given fluids.

Let V_1 V_2 V_3 etc. be the volume of the fluids, let s_1 s_2 s_3 be their specific gravities. Then the weight of the mixture is $w\ (s_1 V_1 + s_2 V_2 + s_3 V_3 + \text{etc.})$

$\therefore$ Weight of same volume of water is $w\ (V_1+V_2+V_3+\text{etc.})$, and consequently the specific gravity of the mixture is

$$\frac{s_1 V_1 + s_2 V_2 + s_3 V_3 + \text{etc.}}{V_1 + V_2 + V_3 + \text{etc.}}$$

Example 7.—To find the specific gravity of a mixture where the weight and specific gravity of the compounds are known.

Let w_1 w_2 w_3 etc., be the weights, let s_1 s_2 s_3 be the specific gravities.

The volumes are $\frac{w_1}{s_1}$, $\frac{w_2}{s_2}$ etc. Hence, substituting in the last exercise, we have the specific gravity of the mixture

$$=\frac{w_1 + w_2 + w_3 + \text{etc.}}{\frac{w_1}{s_1} + \frac{w_2}{s_2} + \frac{w_3}{s_3} + \text{etc.}}$$

EXERCISES.

1. A body measuring 18 cubic centimetres floats in water with its whole bulk immersed; find its weight.

2. Define the specific gravity of a body. If 150 cubic centimetres of a body weigh 120 grammes; find its specific gravity.—*Ans.* ·8.

3. If 200 cubic centimetres of brass have a specific gravity of 7·5, find the weight.—*Ans.* 1500 grammes.

4. Having given that 10 cubic feet of stone weighing 24,950 ozs., have a specific gravity of 2·5, find the weight of a cubic foot of water.—*Ans.* 998 ozs.

5. What is the weight of 12 cubic inches of mercury, the specific gravity of which is 13·5, if a cubic foot of water weighs 1000 ozs.—*Ans.* 93·75 ozs.

6. What volume of zinc, of specific gravity 6·85, will weigh 109·6 grammes?—*Ans.* 16 cubic centimetres.

7. What is the size of a block of stone, the weight of which is 1 cwt., and specific gravity 2·4?—*Ans.* ·746 cubic feet.

8. Three liquids, the specific gravities of which are respectively 1·2, ·96, and 1·456, are mixed in the proportions of $1\frac{1}{2}$ gals. of the first to $1\frac{1}{3}$ gals. of the second, and $1\frac{1}{4}$ gals. of the third; find the specific gravity of the mixture.—*Ans.* 1·2.

9. What will be the weight in water of 100 cubic centimetres of iron, the true weight being 784 grammes.—*Ans.* 684 grammes.

10. A cubic centimetre of alcohol weighs ·79 grammes, what will be the true weight of a body which floats in alcohol with 1000 cubic centimetres of its volume immersed?—*Ans.* 790 grammes.

11. What is the weight of a cubic centimetre of a body which floats in water with one-fifth of its volume above the surface?—*Ans.* ·8 grammes.

12. Find the weight of a cubic foot of mercury, the specific gravity of which is 13·568, supposing a cubic foot of water to weigh 1000 ozs.—*Ans.* 848 lbs.

13. If a cubic unit of the standard substance weigh ·85, what is the weight of 1000 cubic units of substance whose specific gravity is 3?—*Ans.* 1050.

14. The specific gravity of two liquids are respectively 1·3 and

·85. A measure of the former is added to three measures of the latter; find the specific gravity of the mixture.—*Ans.* ·9625.

15. Ten cubic centimetres of a substance weighs $\frac{1}{16}$ kilogs.; find the specific gravity of the substance.—*Ans.* 6·25.

16. Equal volumes of three fluids are mixed, the specific gravity of the first is 1·55, that of the second is 1·75, and that of the mixture is 1·6; find the specific gravity of the third.—*Ans.* 1·5.

VI.—TO FIND THE SPECIFIC GRAVITY.

33.—To find the relative weight or specific gravity of a body, we require to know its weight and the weight of an equal bulk of water. We will now point out methods of obtaining the weight of water equal in bulk to a given body. We have seen that when a solid is immersed in a fluid, it displaces its own volume of the fluid. Suppose that we immerse the body gently in a vessel exactly filled with water, and weigh the water which overflows, we shall then have the required weight of water.

34. A more convenient method is afforded by the principle that the solid, when weighed in a fluid, appears to be lighter by the weight of the fluid displaced. This law is termed the principle of Archimedes. It is said that Hiero, King of Syracuse, applied to Archimedes for a test to prove whether a crown which had been made by his orders

was all gold, or whether the goldsmith had dishonestly substituted a baser metal for a portion of the gold. While the philosopher was thinking of the subject, he chanced to enter a bath filled with water, and noticed that, as he entered, the liquid flowed over.

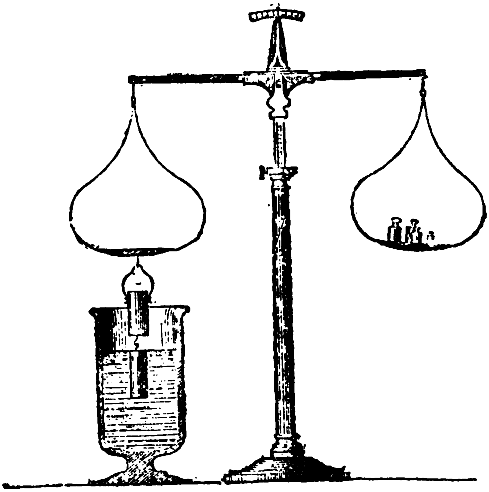

Fig. 13.

This observation suggested a solution of his problem. He took the crown, and a quantity of pure gold of the same weight, and immersed them successively in the same vessel, filled to the brim with water. As the crown displaced more water than the equal bulk of *gold, he* concluded that it was partly composed of a lighter metal, and the king's suspicions were confirmed.

35. The principle of Archimedes may be tested experimentally thus:—Two cylinders, the first *a* hollow, and the second *b* solid, and of such a size as exactly to fill the first, are suspended from the scale of a balance and weighed. The cylinder *b* is then suspended from *a* and allowed to dip into a vessel of water. The opposite scale then goes down; but it is found that equilibrium is restored when the cylinder *a* is filled with water.

A Solid heavier than Water.

36. Hence, to find the specific gravity of a solid heavier than water, and not soluble in it, weigh the body in air, then weigh it in water; the specific gravity

$$=\frac{\text{the true weight}}{\text{loss of weight in water.}}$$

Example.—A piece of metal weighs 18 kilogs. in air, and 15 in water; find its specific gravity.

The loss of weight in water$=18-15=3$ kilogs.

Hence the specific gravity $=\frac{18}{3}=6$.

37. An instrument termed Nicholson's hydrometer may be employed to compare the weight of a body with that of an equal bulk of fluid.

It consists of a hollow cylinder *c* (Fig 14), supporting a light cup *b* on a thin stem above it, and a heavy cup *h* below it. The whole floats in the fluid. A certain weight (suppose 10 grammes) is placed in the upper cup, and a mark is made on the stem, to indicate the surface of the liquid. The body is then placed in the cup; the hydrometer will descend, in consequence of the additional weight. Such weights are then taken off, as will cause the instrument to rise until the mark made is again at the surface. The weights taken off will be the weight of the body. For example, let us suppose 7 grammes removed, then this is the weight of the body. Now remove the body to the lower cup; the instrument ascends. Place such a weight on it as will bring it to the same level. The weight added is the loss of weight in water. Suppose 3 grammes added, then the specific gravity

$$= \frac{7}{3} = 2\frac{1}{3}.$$

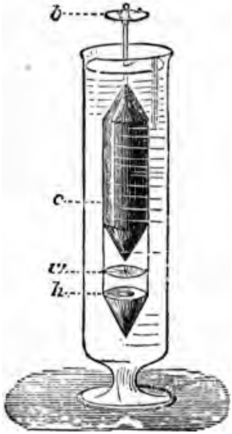

Fig. 14.

To Find the Specific Gravity of a Solid Lighter than Water.

38. 1*st Method.*—If a wire gauze *w* be stretched *over the lower* cup in Nicholson's hydrometer, so as

to prevent a light body rising, the method last described will suit this case also.

2nd Method.—The specific gravity of a solid lighter than water may also be found by attaching a heavier body to it.

Example.—A block of wood, weighing in air 8 lbs., is tied to a piece of iron weighing 6 lbs.; the whole weighs in water 4 lbs., and the iron only weighs 5 lbs. Find the specific gravity of the wood.

The weight of water equal in bulk to the iron . $=6-5=1$

The weight of water equal in bulk to both iron and wood$=(8+6)-4=10$

$\therefore$ Weight of water equal in bulk to the wood . $=10-1=9.$

Hence the specific gravity of the wood$=\frac{8}{9}$. And in every case the specific gravity of the lighter body

$$=\frac{\text{weight of light body}}{\text{loss of weight of both}-\text{loss of weight of heavy body}}$$

To find the Specific Gravity of a Liquid.

39. *1st Method.*—Weigh a flask filled with the liquid, then weigh the flask filled with water; deduct from each result the weight of the flask, then the specific gravity of the liquid is its weight, divided by the weight of the water.

Example.—A flask, when empty, weighs 2 ozs., when filled with alcohol it weighs 6 ozs., and when filled with water it weighs 7 ozs. Find the specific gravity of the alcohol.

The weight of the alcohol is $6-2=4$ ozs.

The weight of the water is $7-2=5$ ozs.

∴ The specific gravity of the alcohol $=4\div 5=\cdot 8$.

2nd Method.—Weigh a body of known weight in the given liquid and in water.

The weight of the body's bulk of the liquid is the loss of weight in the liquid.

The weight of the body's bulk of water is the loss of weight in water.

Hence the specific gravity

$$=\frac{\text{the loss of weight in the liquid}}{\text{the loss of weight in water.}}$$

Example.—A platinum ball, weighing 2 lbs., when suspended in olive oil, weighs 1·915, and in water weighs 1·907. Find the specific gravity of the oil.

Weight of oil displaced $=2-1\cdot 915=\cdot 085$

Weight of water „ $=2-1\cdot 907=\cdot 093$

∴ Sp. gr. of the oil $=\cdot 085\div\cdot 093=\cdot 915$.

To find the Specific Gravity of a Gas.

40. Use a flask with an air-tight stop-cock, exhaust it by means of an air-pump, weigh it first empty, then full of the gas under consideration, and, finally, full of water. On deducting the first result from each of the other two, we have specific gravity of gas

$$=\frac{\text{weight of gas contained by the flask}}{\text{weight of water contained by the flask.}}$$

Example.—A flask weighing 10 grms. when empty, weighs 10·6465 grms. when filled with air, and 510 grms. when filled with water. Find the relative weight of air.

The air which fills the flask weighs ·6465 grms.

And the water 500 grms.

The relative weight is therefore

$$\cdot 6465 \div 500 = \cdot 001293.$$

To find the Specific Gravity of a Solid Body Soluble in Water.

41. Weigh the body in air, and then in some liquid in which it is insoluble. The loss of weight is the weight of a quantity of this liquid having the same bulk as the body. The weight of the same bulk of water is found by dividing the result by the specific gravity of the liquid.

Example.—A piece of sodium weighs 8 grms. in air, and 1·07 grms. in naphtha, the specific gravity of which is known to be ·84. Find the specific gravity of the sodium.

The weight of the naphtha having the same volume as the sodium $= 8 - 1{\cdot}07$, or 6·93 grms.

The weight of the same bulk of water =

$$6{\cdot}93 \div {\cdot}84, \text{ or } 8{\cdot}25.$$

Hence the specific gravity $= \dfrac{8}{8{\cdot}25} = {\cdot}97.$

Instead of definite weights, let us take general symbols. Let w_a=the weight of the sodium in air, w_n=the weight in naphtha, and let S=the specific gravity of the naphtha; then

$$\text{the specific gravity}=\frac{w_a}{(w_a-w_n)\div S}=\frac{w_a\cdot S}{w_a-w_n}.$$

Hence we may find the specific gravity referred to water by finding first the specific gravity referred to some other liquid, and then multiplying by the specific gravity of this liquid.

42. We may also find the specific gravity of a liquid by comparing the amount displaced by a body made to float in the liquid and in water. Fig. 15 represents such a body, termed a common hydrometer. It is usually made of glass, and consists of a tube A, a large bulb B, and a small one C, loaded so as to keep the stem vertical.

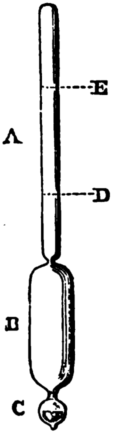

Fig. 15.

Let w be the weight of the instrument, and a the area of a section of the stem. Suppose the instrument to sink to D in water, and to E in the other liquid, then if v be the volume below D, and S the specific gravity of the fluid, $v \div S$ is the volume below E. But since the volume of the part DE is a . ED :

$$\therefore \frac{v}{S}=v+a\,.\,ED.$$

43. In the preceding examples we have not taken account of the weight of the air displaced by the solid weighed in it. The true weight of the solid is the weight in air, together with the weight of the air displaced. We may find approximately the weight of the air displaced from the weight of the water displaced. At the ordinary pressure of the atmosphere 1 litre of water weighs 1000 grms., and 1 litre of air 1·293 grms.; hence the weight of air displaced is ·001293 of the water displaced.

Example.—A body weighs 8 kilogs. in air, and 5 in water; find its specific gravity, taking into account the weight of the air. The weight of water displaced is very nearly 8 — 5, or 3 kilogs. This is not quite true, because 8 is not quite the true weight of the body. Hence the weight of the air is very nearly ·001293 × 3000 grms., or 3·879 grms. Hence the specific gravity is

$$\frac{8003·879}{3003·879}=2·664545.$$

Density.

44. The density of a body is a term for the ratio of the quantity of matter in the body to the quantity of matter in an equal bulk of a standard substance. Thus, if water be the standard, the density

$$=\left(\frac{\text{mass of the body}}{\text{mass of an equal bulk of water.}}\right)$$

The density, therefore, refers to the mass as the specific gravity refers to the weight of a body.

If we suppose the force of gravity constant, as is the case at any one particular spot on the earth's surface, the weight will be proportional to the mass, and the density and specific gravity of a body will be the same. The variation of the force of gravity does not affect the density, but it does affect the specific gravity.

From the above formula we see that
if the volume be kept the same
the density varies as the mass,
and if the mass remain the same
the density varies inversely as the volume.
Hence, when both volume and mass vary,

$$\text{the density varies as } \frac{\text{mass}}{\text{volume}}$$

Therefore $d = \frac{C\,M}{V}$ where c is a constant.

It has been shown in "Mechanics" (p. 176) that writing M for mass, W for weight, and g for acceleration due to gravity,

W varies as M when g is constant,
W ,, g M ,,
∴ W ,, Mg both vary

$$\therefore W = CMg,$$

but from the above formula

$$M = dV \times \text{a constant}$$

$$\therefore W = cdgV$$

The units may be so chosen, therefore, that

$$W = gdV.$$

Problems on the Density of Liquids.

45. If two liquids that do not mix meet in a bent tube, their heights above their common surface will be inversely proportional to their densities. Let A be the upper surface of the denser fluid of density d, B that of the lighter fluid of density d_1, and let C be their common surface (Fig. 16). Let h = the height of A above the horizontal plane C D and h_1 = height of B above C D.

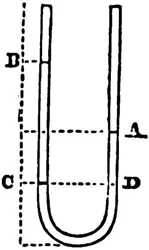

Fig. 16.

The pressure at C = gd_1h_1 the pressure at D = gdh, and these are equal, since they are in the same horizontal plane.

$$\therefore\ gdh = gd_1h_1$$

$$\text{or } h : h_1 = \frac{1}{d} : \frac{1}{d_1}.$$

Examples.

1. A body suspended by a thread from one scale of a balance weighs 36·84 grammes; a vessel of water is placed so that the body is suspended in the water, and then 6·21 grammes must be placed in the scale above the body to restore equilibrium; find the specific gravity of the body.—*Ans.* 5·9.

2. Find the specific gravity of a piece of lead which weighs 47·48 grammes in air and 43·33 grammes in water.

3. The specific gravity of aluminium is 2·56, and a fragment when weighed in water loses 3·78 grammes. What is its weight? —*Ans.* 9·6768 grammes.

4. A body weighs 5 grammes; when the body and 100 grammes of lead of specific gravity 11·44 are weighed together in water, the loss of weight is 9·5 grammes. What is the specific gravity of the body?—*Ans.* 6·59.

5. A cannon made of metal of specific gravity 8·55 weighs 216 kilogs. in air, and 210 kilogs. in water. Is there a flaw in the cannon? What is its volume?—*Ans.* There is a flaw; the volume = 30,000 cubic centimetres.

6. A body weighs 25 grammes in air, 20 grammes in water, and 15 grammes in another liquid; find the specific gravity of the body and of the second liquid.—*Ans.* 5 and 2.

7. A balloon weighs 263·525 grammes when empty, and 379·835 grammes when filled with air at the ordinary pressure. What is its capacity, the weight of a litre of the air being 1·293 grammes?—*Ans.* 90 litres.

8. The same balloon, when filled with gas under the same conditions of temperature and pressure, weighs 293·687. What is the specific gravity of the gas referred to air as the standard?—*Ans.* ·26 nearly.

9. A body floats with one-third of its bulk under water; what is its specific gravity?

Let W = weight of body; then ⅓ of its bulk of water = W, and the weight of its own bulk of water is therefore 3 W. The specific gravity is therefore W ÷ 3 W = ⅓.

10. A piece of gold weighs 9·7 grammes, a flask full of water weighs 95 grammes; the gold is put into the flask, and displaces a small quantity of the water, which flows over. The weight of the whole is 104·2 grammes; find the specific gravity of the gold.—*Ans.* 19·4.

11. Nicholson's hydrometer is used as follows:—51·72 grammes is placed on the upper cup, a fragment of metal is placed on the cup, and it is found necessary to take off 14·85 grammes to make the instrument float at the same level; the metal is then placed in the lower cup, and 2·03 grammes added above to restore the former level. Find the specific gravity of the metal.—*Ans.* 7·31.

12. A globe of glass weighs in pure water 6·9 grammes, in sea water 6·2 grammes, and in air 13 grammes. Find the specific gravity of the sea water.—*Ans.* 1·115.

13. A cylindrical piece of cork of specific gravity ·25 and length of 10 inches floats upright in water; how much of it will be immersed?—*Ans.* 2·5 inches.

14. A piece of wood weighs 7 lbs. and a piece of iron weighs 7·8 lbs. in air and 6·7 lbs. in water; the wood and iron together weigh 5·3 lbs. in water. What is the specific gravity of the wood?—*Ans.* ⅚.

15. A body weighs 10 ozs. more in air than in water; if 1 cubic foot of water weighs 1000 ozs., what is the volume of the body in cubic feet?—*Ans.* ·01.

16. A cylinder floats in a fluid of specific gravity 2·5 with two-thirds of its bulk above the surface. What is the specific gravity of the cylinder?—*Ans.* ·83.

(The specific gravity referred to the liquid $= \frac{1}{3} \therefore$ referred to water $= \frac{1}{3} \times 2·5$.)

17. If the cylinder be removed to another fluid, and then floats with one-third of its bulk out, find the specific gravity of the fluid. —*Ans.* 1·25.

18. A vessel containing water is placed in one scale of a balance, and counterpoised by a weight placed in the other scale. If 10 cubic centimetres of metal be suspended in the water so that none of the water is spilt, what additional weight must be placed in the other scale to restore equilibrium?—*Ans.* 10 grammes.

19. To one scale of a balance a weight of 20 lbs. and specific gravity 8·5 is suspended, and allowed to rest in water, what weights must be put in the other scale to restore equilibrium?

Solution. 8·5 lbs. of metal lose 1 lb. in water.

„ „ weigh 7·5 lbs. in water.

1 lb. „ weighs $\frac{7·5}{8·5}$ „

20 lbs. „ „ $\frac{20 \times 7·5}{8·5} = 17\frac{11}{17}$

20. A rod of wood floats with one-fourth of its bulk out of water, and a stone equal in bulk to one-tenth of the rod, when placed on the rod, sinks it until the top of the rod is level with the surface. Find the specific gravity of the stone.—*Ans.* 2·5.

21. A body whose specific gravity is unity is placed in water, how much weight does it lose?

22. A vessel and water weigh 10 lbs., what will they weigh when a body which floats with 5·76 cubic inches immersed is placed in the water.—*Ans.* $10\frac{5}{24}$ lbs.

23. Two bodies, *a* and *b*, of different bulks weigh the same in water, which will weigh the most in mercury?—*Ans.* The smaller.

24. Which will weigh the most in air?—*Ans.* The larger.

VII.—THE AIR AND GASES.

46. Many of the properties of liquids are also possessed by elastic fluids. For example, the air has weight. To prove this, let us take a flask fitted with an air-tight stop-cock, and by means of an instrument to be hereafter described, let us pump all the air out of it, and turn off the stop-cock. Let the flask be placed in the scale of a balance, and such weights in the other as will keep the beam horizontal. When the stop-cock is turned on, the air will be heard to rush in and the flask will descend. If, instead of the flask, a glass globe capable of containing a cubic foot be emptied and weighed, when the air is admitted it will be an ounce and a half heavier than before. Again, pressure is transmitted by liquids and gases according to the same laws; for example, the pressure at a point in a gas is the same in all directions.

47. A very simple experiment will prove the existence of atmospheric pressure. Fill a tumbler with water, and place a sheet of paper over the top, hold the palm of the hand against the paper, and invert the tumbler. The hand may now be withdrawn without the water falling (Fig. 17). The atmospheric pressure supports the water, the paper being used only to prevent the air from replacing them in the glass.

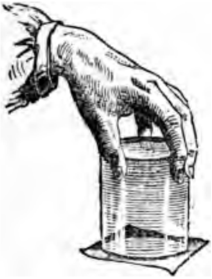

Fig. 17.

48. If a glass cylinder be tightly closed at one end by an air-tight membrane, and the air be withdrawn from the interior by means of an air-pump, the pressure of the atmosphere, not being supported by the elastic force of the air inside the cylinder will burst the membrane.

49. Nothing in the history of science is more remarkable than that up to the time of Galileo (1636) the fact that the atmosphere exerted pressure was unknown. Effects produced by this pressure were observed on all sides, but they were set down to other causes. When the water rose in a pipe from which the air had been withdrawn, as is the case with the common pump, in apparent violation of the laws of gravity and of the law that liquids maintain their levels, philosophers, unable to account for the circumstance, satisfied themselves by considering it due to a freak of nature. They set it down as an axiom that "nature abhors a vacuum," that nature would not allow any part of the universe to be void of matter. Thus, they argued, when air has been withdrawn from the pipe of a pump, nature's abhorrence of a vacuum compels the water to rise and fill the vacant space.

In 1636, however, some mechanics made a pump to raise water from a well, the surface of the water being 50 feet deep. They found that they could not make the water rise more than 32 feet. They applied to the celebrated philosopher, Galileo, to solve the mystery. He could not explain it, but he said one thing was certain—Nature's abhorrence of a vacuum did not extend over 32 feet.

The subject was then investigated by Torricelli, a pupil of Galileo. He argued that whatever the cause might be, since it was sufficient to support 32 feet of water, if a heavier liquid were used, a column of less altitude should be supported. For example, since 32 feet of water is sustained, of mercury, which is 13½ times as heavy as water, 32 feet divided by 13½, or 28 inches, ought to be supported.

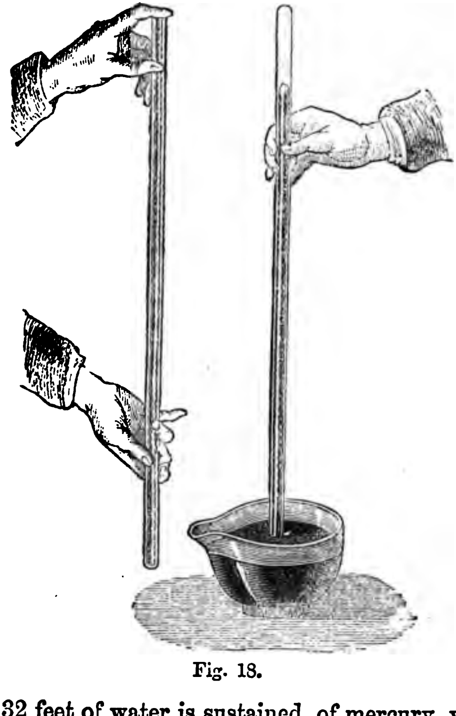

Fig. 18.

He conducted his experiment as follows:—He took a vessel of mercury and a glass tube over 30 inches long, open at one end and closed at the other. Having filled the tube with mercury (Fig. 18), he closed the end with his finger, inverted the tube, plunged the covered end below the surface of the mercury in the vessel, and removed his hand.

The mercury in the tube subsided until it stood at the height of 28 inches.

Supposing the surface of the mercury in the vessel to be prolonged under the tube, he saw that the fluid outside the tube was only subjected to the pressure of the atmosphere, while that under the tube sustained the column of mercury above it. The secret was now revealed to him, for he concluded at once that the pressure of the atmosphere on an area equal to the aperture of the tube was equal to the weight of the mercury in the tube.

This conclusion was confirmed by carrying the apparatus up a mountain. Pascal, a French divine, argued that when a portion of the air was left below, the pressure of that which remained must be diminished; consequently, if the column of mercury were really supported by this pressure, the height of the column must vary with different distances above the earth's surface. The experiment was tried, and the mercury in the tube was found gradually to sink during the ascent.

Magdeburg Hemispheres.

50. A few years later (in 1654) a Professor in Magdeburg constructed an apparatus which not only

exhibited the force of the atmospheric pressure, but showed that this pressure acted equally in all directions. The apparatus consists of two hollow metallic hemispheres, the edges of which are made to fit exactly (Fig. 19). To one of the hemispheres there is a tube and stop-cock, so that when the two are placed together the air may be pumped out. When filled with air the hemispheres may be separated without difficulty, but, when the air is exhausted, in whatever position they are held, a powerful effort is required to separate them.

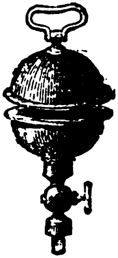

Fig. 19.

The Barometer.

51. The apparatus used in Torricelli's experiment constitutes a barometer. Whatever the size of the tube, the pressure at the base of the column on a level with the surface of the liquid outside must be equal to the pressure of the air on the same area anywhere in that surface.

Thus, if the area of a section of the tube be one square inch, the pressure on a square inch is the weight of the mercury in the tube; if the section be a tenth of a square inch, the weight of the column of mercury above the surface is the pressure of the air on the tenth of a square inch, and ten times this weight is, of course, the pressure on a square inch.

It is easy, therefore, to find by means of this experiment the pressure of the air. Suppose, for *example, a tube* is used having a section exactly one square centimetre in area, and suppose the column of mercury to stand at the usual height of 76 centimetres; there will then be 76 cubic centimetres of mercury in the tube. Now, the specific gravity of mercury is 13·596, hence 76 cubic centimetres of mercury weigh 13·596 grammes × 76 = 1033 grammes. A pressure of 1033 grammes on a square centimetre is termed a pressure of one atmosphere.

If we take inches and ounces as the units, supposing the height of the barometer to be 30 inches, and the weight of a cubic inch of water $\frac{1000}{1728}$ ozs., the atmospheric pressure $= 30 \times 13{\cdot}596 \times 1000 \div 1728$ ozs. $= 14{\cdot}7$ lbs.

Marriotte's Law.

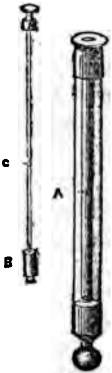

Fig. 20.

52. On account of the elasticity of a gas, the pressure exerted by it changes with its temperature and with its volume.

For example, take a cylindrical vessel fitted with an air-tight piston (Fig. 20). When held vertical, the piston will be supported by the resistance of the air within the cylinder. An effort is required to force in the piston, and the more the air is compressed the greater the pressure required to keep the piston from returning.

Instead of a force applied by the hand, let us place a weight on the piston. Suppose the pressure of the air as indicated by the barometer to be 15 lbs. on the square inch. Suppose, for convenience of reckoning, we take a cylinder A B, of 1 square inch in area, and 12 inches long. Before the piston is inserted the air in the cylinder sustains a pressure of 15 lbs. Put in the piston, and place on it such weights as will cause it to descend to A, the middle of the cylinder. The air will have been compressed into half its bulk, and it will be found that the weight amounts to 15 lbs. Then double the pressure reduces the volume one half. If we make the piston and its load twice 15 lbs., so that the whole pressure is thrice 15 lbs., the piston will descend to E, one-third of the height. When the pressure is four times 15 lbs., the bulk is one-fourth of that at a pressure of 15 lbs. A pressure of 12 atmospheres will reduce the air to one-twelfth of its original volume.

This relation of the pressure to the volume of a gas is termed Marriotte's Law.

"If the temperature be kept the same, the pressure of a quantity of air or gas is inversely proportional to the space it occupies."

This law was proved by Boyle and Marriotte as follows:—A bent tube A B with open ends (Fig. 21), and one arm A longer than the other, is fixed to a graduated stand, and a little mercury is poured into it. The surface C C′ of the mercury will be in the same horizontal plane, and will be subjected in both tubes to the pressure of the atmosphere. The end B is now closed, and more mercury poured into A. The *mercury rises to* D in the arm B, compressing

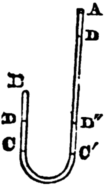

Fig. 21.

the air, and to D′ in the arm A. The pressure of the mercury in the two arms below the same level D D″ balance each other, and the pressure at D″ is therefore that of the atmosphere, added to the weight of the column D′ D″.

Suppose P_1 be written for the original pressure on the air in B, and P_2 for the pressure after compression; suppose that the pressure of the air would support the mercury at a height h, and let the weight of mercury in one inch of the tube be w,

then the pressure of the atmosphere $= hw$,

and that of the atmosphere and the column $D'D'' = (h + D'D'')w$,

hence $\dfrac{P_1}{P_2} = \dfrac{h}{h + D'D''}$

Now, when the scale is applied to the spaces BC, BD, it is always found that $\dfrac{BD}{BC} = \dfrac{h}{h + D'D'}$,

hence $\dfrac{BD}{BC} = \dfrac{P_1}{P_2}$

The relation between pressure and volume may be also expressed thus:—If the temperature remains the same, the volume multiplied by the pressure remains the same. Thus, if V be the volume when the pressure is P, and V^1 the volume when the pressure is P^1, $VP = V^1P^1$.

Relation of Pressure and Temperature.

53. The pressure of a gas is affected by change of temperature.

If the heat syringe (Fig. 20), with a certain load on the piston, be placed in warm water, the piston will be forced up unless additional weight be added.

If the pressure remain the same, an increase of temperature of 1° centigrade produces in a given mass of gas an expansion of ·003663 of the volume it would have at 0° centigrade.*

The Weather-Glass.

54. The pressure of the atmosphere not only varies with the height above the sea, but it changes at the same place within certain limits; consequently, the height of the mercury in the barometer-tube is subjected to frequent variations. These variations at the level of the sea range from 785 to 731 millimetres, or from 31 to 28 inches.

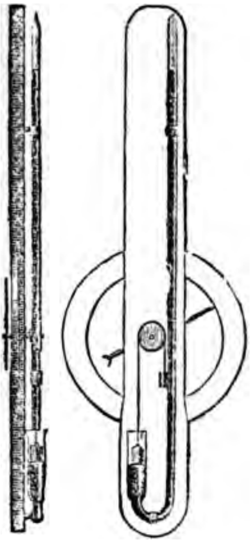

Fig. 22.

A change in the height of the barometer frequently coincides with a change of the weather. In our climate it is a matter of observation, that in rainy and stormy weather the barometer is usually below the average height, and in fine and dry weather the height of the barometer is usually above the average. Hence the barometer is used as a weather-glass.

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* For explanation and examples of this law, see Orme's *"Heat," page 64.*

The form of instrument used for this purpose is represented in Fig. 22. It consists of a siphon barometer, in the shorter leg of which is a float, which rises and falls with the mercury. The float is connected, by some mechanical contrivance, with a wheel, bearing a needle. When the pressure of the air varies, the sinking or rising of the float causes the needle to move over a graduated scale in front of the case enclosing the barometer.

The Common Pump.

55. The pressure of the atmosphere is taken advantage of for the purpose of raising water above its level.

The common pump is a machine by means of which this is effected. A B is a cylinder, having its lower end closed with a valve B (Fig. 23) opening upwards, and connected by means of a pipe with the water which is to be raised. A piston containing either one or two valves opening upwards, is worked in the cylinder by means of a vertical rod and a handle.

Suppose the piston to be in its lowest position, then, when it is raised, the valve B opens, and a partial vacuum is produced in the cylinder and pipe; and the pressure of the atmosphere without being greater than the pressure within the pipe, the water rises, and it continues to rise until the pressure within and without become equal.

When the piston descends the valve in the piston opens, and the air within the cylinder escapes. When the upward-stroke of the piston is complete, it is again depressed; the water passes through the valve in the piston, and on the next stroke it is discharged at the spout.

If the valves and pipes be perfectly air-tight, the greatest height to which the water could be raised by a common pump, reckoning from the surface of the water in the well to the bottom valve, is 33·75 feet, when the mercurial barometer is 30 inches, for

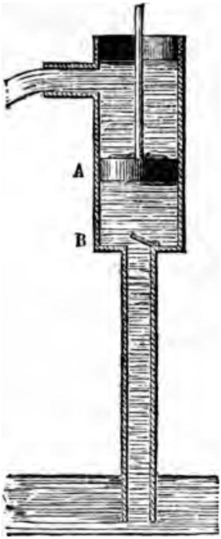

Fig. 23.

30 inches × 13·5 = 33·75 feet,

13·5 being the specific gravity of mercury.

The Lifting Pump.

56. Sometimes pumps are used to lift water to a great height. In this case the spout is connected with a vertical tube, and the water, instead of escaping by the spout, rises in the tube, and is *prevented from* returning by a valve.

The Archimedian Screw.

57. One of the earliest machines used for lifting water on record is the screw of Archimedes (Fig. 24). It consists of a cylinder inclined to the vertical, and exactly fitted by a screw having the same axis. If a small solid body were placed at the bottom, and the screw turned round, each point of the screw would pass beneath the body at the lower edge of the cylinder, and the body would thus be raised gradually to the top. In the same manner, if water has access to the bottom, on turning the instrument it will be gradually raised until it flows out at the top.

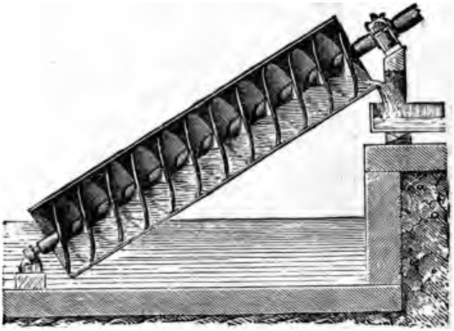

Fig. 24.

The Forcing Pump.

58. The forcing pump differs from the common pump in the following respects: first, there is no spout or outlet to the upper part of the cylinder; secondly, the piston A is solid; thirdly, in addition to the pipe closed by the valve C, and communicating with the cistern, there is another pipe passing from

the lower part of the cylinder, with a valve c′ opening upwards, through which the water is to be raised. ;

Suppose the piston to be in its lowest position when it is raised the valve c opens and c′ closes, the column of water within the cylinder and pipe is relieved of the pressure above A, and the pressure of the atmosphere without being thus greater than that within the pipe, the water within rises, and continues to rise until the pressures within and without are equal. Let the piston be now made to descend, and the process repeated until the water has risen above the top of the pipe; then when A descends, the water in the cylinder, not being able to return on account of the valve c, is forced up the pipe, in which it is retained by the valve c′.

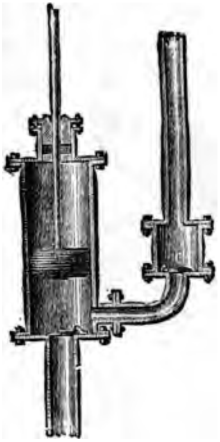

Fig. 25.

The Fire-Engine.

59. When a continuous stream is required, the valve leads into a strong air-vessel, out of which the pipe passes. After the down-stroke of the piston the air in the chamber is condensed, and its pressure on the surface continues to lift the water during the up-stroke. The fire-engine consists of two force-pumps *connected* with an air chamber (Fig. 26). The

pumps are worked by a lever, and are arranged so that when one piston descends and closes the valve below it, the other ascends and opens the valve communicating with the water in the cistern.

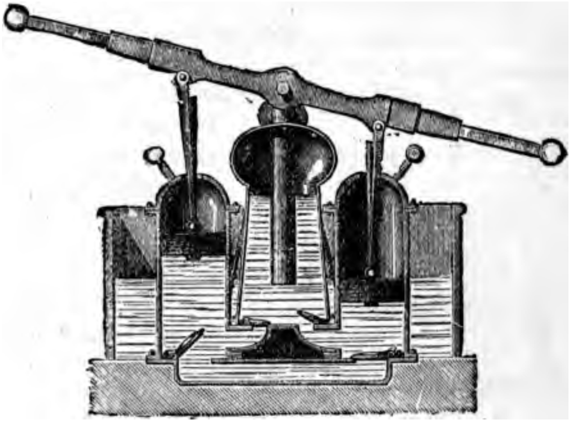

Fig. 26.

The Siphon.

60. The siphon is a bent tube A B C (Fig. 27) open at both ends. Let the tube be filled with fluid, and the shorter leg inserted into a vessel of fluid which it is required to empty, and the extremity C of the other leg closed by the finger. Let the level of the surface of the fluid meet the two legs of the siphon in G and D respectively. Provided the height of B above G D is not greater than that of a column of water, the weight of which is equal to the atmospheric pressure, the water in D B will be supported by the atmospheric pressure. Let P be the atmospheric

pressure, the pressure at D=P, and is transmitted along the tube.

The pressure at B=P—the pressure due to a column of water of height D E.

The pressure at G=P.

The pressure at C=P+the pressure due to a column of water of height *g* C.

Consequently the pressure on the end C, which we have supposed closed, exceeds the atmospheric pressure by the pressure due to a column of water of height of *g* C, or C F—D E.

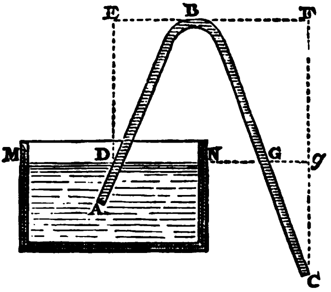

Fig. 27.

If the end C be opened the fluid will descend. Since at every point of the tube there is a pressure in the direction A B C, as soon as the liquid at any point moves forward, the particles behind being relieved of the pressure in front immediately follow, and thus a continuous stream is produced, which will only cease when the surface of the fluid has descended to A, the extremity of the shorter leg of the siphon.

If the leg B C terminated at G, it is evident that there would be equilibrium.

If the height D E were greater than that of the water-barometer, the liquid would descend in each tube to that height, leaving a vacuum at B

The Air-Pump.

61. The air-pump is a machine constructed for removing the air from a closed vessel, called a receiver.

From the centre of a metal plate to which the receiver R is fitted (Fig. 28), so that no air can pass between them, runs a pipe communicating with the barrel of the pump B by means of a valve *e*, opening upwards. In the barrel is an air-tight piston H also furnished with a valve opening upwards.

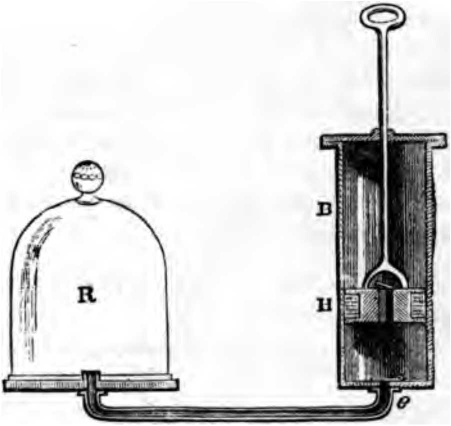

Fig. 28.

Suppose H to be drawn up, then a partial vacuum will be formed below the piston, and fresh air rushing in from R will open the valve at *e*.

Now let H be made to descend, then the pressure of the air will close the valve *e* and open the valve in H, and the air in the barrel will be forced out. In the meantime the air in the receiver expands and refills the barrel.

Now H again descending and ascending expels another portion, and so on till the air in R becomes so rare that its elastic force is insufficient to open the valves. It is obvious that, although the air gets rarer and rarer at each stroke of the piston, the whole can never be exhausted.

The pump which we have described is called the Single-barrelled, or Smeaton's Air-pump. Hawksbee's Air-pump is fitted with two barrels instead of one (Fig. 29). The two pistons are moved by a toothed wheel in such a manner, that one descends while the other ascends, and thus the air can be forced out with greater rapidity.

62. *To find the density of the air in a receiver after any number of turns of the wheel.*

Let R be the volume of the receiver and pipe, and b the volume of each barrel. Then the air which occupied the space R when H was at the bottom of the barrel, fills the space $R + b$, when H ascends to the top.

If therefore $d =$ the density before the stroke, and d_1 the density afterwards, we shall have

$$d_1 (R + b) = d \cdot R.$$

In the same manner it will be found if d_2 be the density after two turns, d_n the density after n turns, that

$$d_2 (R + b) = d_1 \cdot R \qquad \therefore d_2 (R + b)^2 = d \cdot R^2$$
$$d_3 (R + b) = d_2 \cdot R \qquad \therefore d_3 (R + b)^3 = d \cdot R^3$$
$$\therefore d_n (R + b)^n = d \cdot R^n.$$

Thus it appears that the density of the air decreases in a geometrical progression.

The Siphon Gauge.

This is a glass tube s (Fig. 29) screwed to a pipe communicating with the receiver. One end *a* is

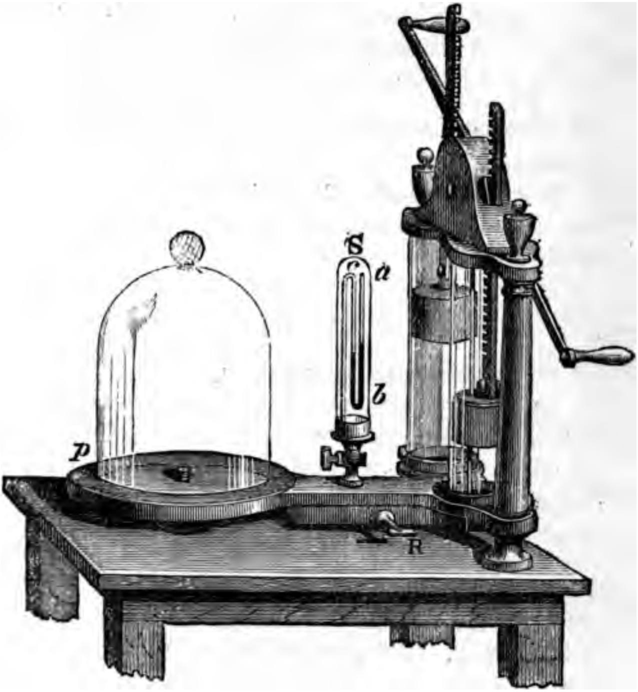

Fig. 29.

closed, and a portion of the tube, rather more than *a b*, completely filled with mercury.

If it be not more than 28 inches in length, the tube *a b* will at first remain filled; but as the exhaustion proceeds, the mercury will sink in *a b* and rise in *b c*, and the difference in the heights of the mercury in *a b* and *b c* will always measure the pressure in the receiver.

Thus, if we call the pressure of the air which supports 76 centimetres of mercury *one atmosphere*, and let n centimetres = the difference in the heights, the pressure of the air in the receiver is $\frac{n}{76}$ths of one atmosphere.

If d be the density of the air outside, and d_1 the density in the receiver, since density is proportional to the pressure,

$$\frac{d_1}{d} = \frac{n}{\text{height of barometer.}}$$

The Diving-Bell.

63. The diving-bell is a strong iron vessel, which is immersed in water with its mouth downwards. The air within prevents the water from filling the vessel; but as it descends the pressure increases, and the air is compressed. Let us suppose that the pressure of the air at the surface is equal to the weight of a column of water 33 feet high, then when the bell has descended 33 feet, the air within will be compressed to half its original bulk. The pressure of the air inside would support a column of mercury twice as high as the mercurial barometer at the surface. At a depth of twic *33 feet* the air would be compressed to one-third i *original bulk,* and so on.

To prevent the advance of the water in the bell, and to supply the workmen with fresh air, a tube, connected with a force-pump at the surface, passes to the bell. There is also a tube through which the workmen can allow the foul air to escape by turning on a stop-cock.

The bell is suspended by chains or ropes.

Exercise.—A cylindrical diving-bell 8 feet long descends until the water in the bell is 80 feet below the surface, when the height of a water-barometer is 32 feet. If no air has been supplied from above, to what height has the water risen in the bell?

The pressure at the surface = the pressure of 32 ft. of water.

" " a depth of 80 ft. = the pressure of 112 ft. of water.

With a pressure of 32 ft. of water, the air occupies 8 ft. of the cylinder.

∴ With a pressure of 1 ft. of water, the air would occupy 8 × 32 ft.;

and with a pressure of 112 ft. of water,

$$\frac{8 \times 32}{112}, \text{ or } 2\tfrac{2}{7} \text{ ft.}$$

EXERCISES.

1. When the height of the barometer is 76 centimetres, find the height to which water will rise in a tube from which the air has been withdrawn, the specific gravity of mercury being 13·596. —*Ans.* 1033 centimetres.

2. What would be the height of the barometer if a liquid of specific gravity 2·5 were used, when the mercurial barometer stands at 76 centimetres?—*Ans.* 413·3 centimetres.

3. When the water barometer stands at 33 feet, what would be

the height if alcohol of specific gravity ·825 were used instead of water?—*Ans.* 40 feet.

4. What would be the effect of making a small aperture at the highest point of a siphon?

5. What difference is made in the height of the column of mercury in a barometer when the tube is inclined?

6. How would the action of a siphon be affected by carrying it up a mountain?

7. If when the barometer stands at 30 inches, the air in the glass cylinder represented in Fig. 28 is half exhausted; find the pressure on a square inch, taking the specific gravity of mercury as 13·5, and the weight of a cubic foot of water as 1000 ozs.

8. If the weight of a cubic inch of mercury be 7·8 ozs., what is the pressure of the air on a square inch when the mercury stands at 29·5 inches?—*Ans.* 230·1 ozs.

9. What would be the height of the water barometer in the above case, the specific gravity of mercury being 13·5?—*Ans.* 398·25 inches, or 33 feet 2¼ inches.

10. A barometer is partly filled with water and partly with mercury, and the height of the water is three times that of the mercury; find the height when the pressure of the atmosphere is 1020 grammes on a square centimetre, the specific gravity of mercury being 13·596.

Solution. Let x be the height of the mercury;
then $3x =$ the height of the water,
and $13·596\,x =$ the height of water equal to x centims. of mercury.

Now, a water-barometer would stand at 1020 centimetres.

$$\therefore\ 3x + 13·596x = 1020$$

$$\therefore\ x = 61·5 \text{ centimetres.}$$

11. How will a fall in the barometer affect the action of a siphon?

12. How will a change in the barometer affect the pressure on the base and sides of a cubical vessel filled with water?

13. Find the greatest height over which a liquid of specific gravity 3·5 can be carried by means of a siphon, when the barometer stands at 760 millimetres.

14. If a vessel of water containing a floating body be placed under the receiver of an air-pump, and the air gradually exhausted, what will be the effect on the floating body?

Solution. The weight of the floating body is equal to the weight of fluid (air and water) displaced; hence, as the surrounding air is withdrawn—that is to say, as the air displaced by the body is diminished—the water displaced must increase. When the receiver is completely exhausted, the body will sink, until the weight of the water displaced is alone equal to the weight of the body.

15. To what height would the mercury rise in a Torricellian tube, if it be immersed in water until the surface of the mercury in the cistern is 45 inches below the surface of the water, supposing the barometer to stand at 30 inches, and the specific gravity of mercury to be 13·5.—*Ans.* 33⅓ inches.

16. The receiver of an air-pump is five times as large as the barrel; after how many strokes will the density of the air be diminished by nearly a half?—*Ans.* 3.

17. In the above example, how much air has been removed after six strokes?—*Ans.* Taking the original volume as unity, the air removed is

$$1-\left(\frac{5}{6}\right)^6.$$

18. What would be the effect of making a small hole in the top of a diving-bell?

19. What is the pressure on a square centimetre in a diving bell, the surface of the water in the bell being 12 metres below the surface above, and the mercurial barometer at the surface standing at 76 centimetres (specific gravity of mercury = 13·596). *Ans.* 2·233296 kilogs.

20. If a block of wood be floating on the surface of the water within a diving-bell, how will it be affected by the descent of the bell?—(See question 14 above.)

LONDON UNIVERSITY EXAMINATION PAPERS.

From January, 1862, *to January,* 1870.

MATRICULATION.

1. Explain the difficulty of opening a lock-gate when the water is at a different level within and without the lock; also, why the force required to open the gate is not proportional directly to the difference of level.

2. The weight of water is 770 times that of air; at what depth in a lake would a bubble of air be compressed to the density of water, supposing the law of Marriotte to hold good throughout for the compression?

Solution.—Let the pressure at the surface be P grammes on a square centimetre, then the pressure, P', at a depth of x centimetres is $(x+P)$ grammes. Let V be the volume of the bubble at the surface, and V' that at depth x, then by § 52 $V\,P=V'\,P'$, but by hypothesis $V=770\,V'$, therefore by substituting for P' and V' we obtain $x=769\,P$ centimetres. If we take feet and ounces as our units $x=144\times{\cdot}769\,P$ feet.

3. A body weighs in air 1000 grains, in water 300 grains, and in another liquid 420 grains; what is the specific gravity of the latter liquid?—*Ans.* ·8285.

4. Sketch the common pump, describing its action, and stating the limitation to which this is subject.

5. A mercurial barometer is lowered into a vessel of water, so that the surface of the water is finally six inches above the cistern of the barometer. What *kind* of change will take place in the reading of the column of the instrument? Give a reason for your reply

6. Draw a diagram representing (in section) a single-barre *air-pump when* the piston is being lowered.

Explain the action of such a pump.

If the receiver hold 64 grains of air, and 8 grains be removed by the first stroke of the piston, how much will be removed by the second stroke?—*Ans.* 7 grains.

7. If a bottle filled with air be tightly corked, and lowered into the ocean, the cork will be forced in at a certain depth. Why is this? and what will take place if the bottle be filled with water instead of air?

8. Explain why a siphon cannot be used to convey water from a lower to a higher level.

9. A solid weighs *in vacuo* 100 grains, in water 85 grains, and in another fluid 88 grains; what is the specific gravity of this fluid?—*Ans.* ·8.

10. Explain the action of the ordinary lifting pump; find the limit of its action; and show that this will be different at the top of a mountain and at the level of the sea.

11. If a barometer were carried down in a diving-bell, what would take place? Give a rough quantitative result.

12. A solid, soluble in water but not in alcohol, weighs 346 grains in air, and 210 in alcohol; find the specific gravity of the solid, that of alcohol being 0·85.—*Ans.* 2·1625.

13. A tapering tube is bent in the middle at an acute angle, and being filled with water is inverted with the two ends completely immersed beneath the surfaces of the water in two vessels, the narrower leg being held vertical, and *its* end being only just below the surface. Point out the mode in which the water will flow; and show when the flow will cease.

14. Describe and explain the barometer; and state on what principle it is used to determine the height of a mountain.

Might it also be used to determine the depth of the sea?

15. Define clearly what is meant by specific gravity. Is there any difference in specific gravity between 4 lbs. of iron and 2 lbs. of the same metal?

A body whose specific gravity is 3·5, weighs 4 lbs. in water. What is its real weight?—*Ans.* 5·6 lbs.

16. Describe, by means of a diagram, the common pump. State the limit to its action and the cause of the same.

17. Explain the action of the siphon. Is there any limit to the height of an embankment over which water may be carried by means of a siphon from a higher to a lower level?

18. If as much additional air were forced into a closed vessel as it previously contained when in communication with the atmosphere, what would be the pressure on a square inch of the internal surface?

19. Describe the forcing-pump, and state why it cannot in general be replaced by a common pump.

20. State precisely what information is gained by knowing the specific gravity of a body.

Show how to find the specific gravity of a solid lighter than water, and not soluble in that liquid.

21. At what depth in a lake is the pressure of the water, including the atmospheric pressure, three times as great as at the depth of 10 feet, on a day when the height of the liquid column in a water-barometer is 33 feet 6 inches?—*Ans.* 97 feet.

22. A lump of beeswax, weighing 2895 grains, is stuck on to a crystal of quartz weighing 795 grains, and the whole, when suspended in water, is found to weigh 390 grains; find the specific gravity of beeswax, that of quartz being 2·65.—*Ans.* ·965.

23. A barometric tube of half an inch internal diameter is filled in the usual way, and the mercury is found to stand at the height of 30 inches. A cubic inch of air having been allowed to pass into the vacuum above the mercury, the column is found to be depressed 5 inches. What was the volume of the original vacuum?

Solution.—The volume of 5 inches of the tube $= \frac{22}{7} \times \left(\frac{1}{4}\right)^2$ = 1 cubic inch nearly.

Let x be the volume of the vacuum,

then $x + 1$ is the volume occupied by the air.

This air exerts a pressure = 5 inches of mercury.

But the air occupied 1 cubic inch when under a pressure of 30 inches of mercury. Hence, by applying the formula of page 49,

$$(x + 1)\,5 = 30$$

$$\therefore\ x = 5.$$

24. A bottle holds 1500 grains of water, and when filled with alcohol it weighs 1708 grains; but when empty it weighs 520 grains; what is the specific gravity of alcohol?—*Ans.* ·792.

25. Describe the construction and action of the common pump.

26. A tube closed at one end is filled with mercury, and the open end dipped below the surface of mercury in an open vessel. If the tube be inclined at an angle of 60° to the vertical, what is

the greatest length the tube can have so as to remain full of mercury, the height of the barometer being at the time 30 inches? —*Ans.* 60 inches.

27. A piece of cupric sulphate weighs 3 ozs. *in vacuo*, and 1·86 ozs. in oil of turpentine; what is the specific gravity of cupric sulphate, that of turpentine being 0·88?—*Ans.* $2\frac{6}{19}$.

28. If the height of the barometer rises from 30 inches to 30·25 inches, what is the *increase* of pressure (in ozs.) upon a square foot?—the weight of a cubic foot of water being taken to be 1000 ozs., and the specific gravity of mercury 13·56.—*Ans.* 282·5 ozs.

29. Describe an experiment which proves that the upward pressure of a fluid on any substance immersed in it, is equal to the weight of the fluid displaced by the substance. Give a sketch showing the arrangement of the apparatus.

30. If a bladder containing 300 cubic inches of air under a pressure equal to that of 30 inches of mercury be sunk 240 feet below the surface of water, the barometer standing at 28·5 inches, to what volume will the air in the bladder be compressed? [Specific gravity of mercury =13·6.]—*Ans.* 37·2 cubic inches.

31. A piece of metal weighs 211·6 grains *in vacuo*, 187·32 grains in water, and 182·37 grains in a solution of sodic chloride; find the specific gravity of the solution.—*Ans.* 1·2.

32. Describe the common barometer, and point out the principle on which its action is based.

What change in the atmospheric pressure on a square inch is indicated by a fall of one inch in the height of the barometric column? (A cubic inch of water weighs 252·7 grains, and the specific gravity of mercury is 13·6.)—*Ans.* A decrease of pressure of 3436·72 grains, or nearly half a pound on the square inch.

33. If two liquids that do not mix meet in a bent tube open at both ends, show that at rest their heights above the common surface of contact are inversely proportional to their specific gravities (§ 45).

34. The specific gravity of mercury is 13·6, and the height of the mercurial barometer is 30 inches. What is the greatest height to which water can be raised by means of the common pump?—*Ans.* 34 feet.

35. Describe and explain the construction and action of the double-barreled air-pump.

36. Describe the forcing-pump, and explain its action. If the area of the piston is 3·5 square inches and a power of 77 lbs. is employed in forcing it down, find the pressure of the air within the air-chamber, and the height to which water can be raised above the piston when the surface of the water in the chamber is 1 foot above the piston. (One cubic foot of water weighs 62·3 lbs.)

37. A vessel in the shape of a pyramid 5 feet high, and with base 4 feet square is filled with water. Find the pressure upon the base and account for it being greater than the total weight of water in the vessel.

38. Describe the barometer-gauge of the air-pump; and explain how the degree of exhaustion of the air in the receiver is determined by it.

OPTICS.

Definitions.

1. Light is the agent which acts upon the organs of vision so as to produce the sensation of sight. The light by which an object is seen always comes from the object to the eye. Bodies are said to be *luminous* when they affect the organs of sight without the aid of other bodies. Thus, the sun, the stars, a candle, a lamp are luminous; trees and stones are non-luminous. The light by which non-luminous bodies are seen is not emitted by them, but is received from some luminous body, and is simply scattered or dispersed by them: if the luminous body disappear they cease to be visible.

2. Light passes through some bodies, and not through others. When the light is transmitted so perfectly that objects can be seen through the intervening body, the latter is said to be transparent. When light passes through a body, but, as in the case of ground glass, not so that objects may be distinguished through it, the body is said to be *translucent*. Bodies through which no light can be transmitted are called *opaque*. The space through which the phenomena of light take place is called a *medium*. The medium may be a vacuum or a transparent substance,

such as air, water, glass. A medium is homogeneous when its chemical composition and density are the same in all its parts.

Propagation of Light.

3. In a homogeneous transparent medium light is propagated in straight lines. If a cylindrical tube be placed in line with a candle, the light may be seen when the tube is straight, but not when the tube is curved. The same is true if the ends of the tube be closed by glass plates and the tube be filled with water.

Ray—Pencil—Focus.

4. A *ray* is the smallest conceivable portion of light, and is represented by the straight line along which it is supposed to be propagated. A collection of rays is called a *Pencil of Light.* A parallel pencil is one whose component rays are parallel to one another. A converging pencil is one in which the rays all proceed to a single point, and a diverging pencil one in which the rays all proceed from the same point. The point to or from which the rays of a pencil proceed is called the *focus* of the pencil.

Velocity of Light.

5. The velocity with which light is propagated is so great that to ordinary observers on the surface of the earth the interval taken by light to pass from one

point to another is inappreciable, yet by means of astronomical observations, as well as by elaborate and delicate experiments, this velocity has been determined. Light travels from the sun to the earth in 8 minutes 16 seconds. The velocity varies with the medium. Thus, it is not the same in water as in air.

Light incident on a Surface.

6. When a ray of light meets a surface at a point, and the perpendicular (or as it is sometimes termed the *normal*) to the surface is drawn from the point, the angle between the ray and the perpendicular is termed the *angle of incidence.* When rays come in contact with an opaque and polished surface, their direction is changed, and the rays are bent back or *reflected.* The angle between the ray after reflection and the perpendicular to the surface is termed *the angle of reflection.* When a luminous ray meets the surface of a transparent medium it passes into the medium, but not along the same straight line. It is bent or *refracted* at the surface. The angle which the ray after refraction makes with the perpendicular to the surface is termed the *angle of refraction.* Usually reflection and refraction occur together, for when light falls on the surface of a body some of it is scattered or dispersed, making the body visible from all sides, although the light may reach it only in one direction. A portion of the light is absorbed, another portion is reflected, and if the body be transparent a large portion passes into it. We shall, however, consider only the reflection and refrac-

tion of light, and shall suppose polished surfaces to reflect all the light incident on them, and transparent surfaces to transmit all.

Law of Reflection.

7. Light is reflected according to the following laws:—

i. *The incident ray, the reflected ray, and the perpendicular to the surface at the point of incidence lie in the same plane.*

ii. *The angle of incidence is equal to the angle of refraction*, or, in other words, the perpendicular to the surface bisects the angle between the incident and reflected rays. Thus, if A B be a ray incident at B on the surface of which M N is a section, and B D the normal at B, then A B, B D, and B C lie in the same plane, and the angle A B D is equal to the angle D B C.

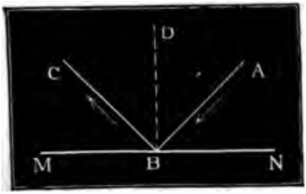

Fig. 1.

Experiment to illustrate the Law of Reflection.

8. To prove the law experimentally, a graduated circle, of which the plane is vertical, is mounted on a stand. Two tubes, movable round the centre, and having their axes directed along radii of the circle, slide

on the arc, and are capable of being adjusted at any angle to the vertical through the highest point A. At the centre of the circle is placed a small plane mirror M, which is exactly perpendicular to the plane of the circle. In making an experiment, a pencil of solar light S is directed along the tube to the mirror M. Here the luminous pencil undergoes reflection, and takes a direction M E, which is ascertained by moving the tube E until the light can be seen as a bright spot in the mirror M, by looking along the tube. On reading off the numbers of degrees in the arcs A E and A S, we find that these arcs are equal. Hence the angle of reflection A M E is equal to the angle of incidence A M S. The first part of the law is proved by the arrangement of the apparatus, since the plane of the rays S M and M E is parallel to that of the graduated circle, and consequently perpendicular to the plane of the mirror M.

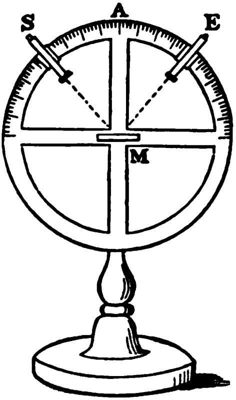

Fig. 2.

MIRRORS.

9. Bodies having polished surfaces in which objects can be seen by reflection are called mirrors. Mirrors are divided, according to their form, into plane, concave, convex, spherical, parabolic, &c.

Images.

10. Before discussing the properties of mirrors, it is necessary to consider what is meant by the image of an object. If an object be placed before an opaque screen having a small aperture about the size of a pinhole in it, and a sheet of paper be held at a short distance beyond the screen, a picture or image of the object will be seen on the paper. Suppose the screen and paper to form opposite sides of a box, as in the figure, then the rays from the upper end of the object will pass straight through the hole, and will illuminate a point A on the back of the box with their own colour; the rays from the lower extremity of the object will do the same at a point B above the former, and rays from intermediate points will fall in a similar manner between A and B. When the hole is small the rays from any one point of the object can fall only on the point of the screen which is opposite to it, and hence the smaller we make the aperture the more distinct will be the picture of the object. If we enlarge the small hole the picture will become more indistinct, and the colours fainter, and when the hole reaches a certain size, the light from different points of the object will

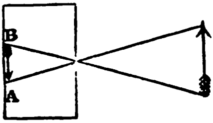

Fig. 3.

fall on the same part of the card, and no image will be formed. Images in every way similar to the above, but much brighter and more distinct, may be produced with a larger aperture by means of mirrors and lenses, the nature of which we proceed to explain.

To explain the formation of images with a plane mirror.

11. When objects are placed before a plane mirror, images of the objects are produced by the reflection of the light which passes from the objects to the mirror.

Let O be a point in the object, and A B a right section of the mirror by a plane through O. Luminous rays proceed from this point in all directions, and some of them fall on the mirror. Let us select one of these, O M; let M N be the normal at M, and M R the direction of the reflected ray. Draw O I from O, perpendicular to the mirror, and continue R M to meet O I in I. Because M N is parallel to O I, the angle R M N is equal to the angle I, and O M N to the angle O. But by the law of reflection R M N = O M N, therefore, also, I = O, and the triangles M O O, M I O, having a common side and the angles of the one respectively equal to the angles of the other, are equal; consequently O O = I O. Thus the ray M R proceeds as if it came from a point I at the back of the mirror, at the same distance from it as the point O, and in the same perpendicular.

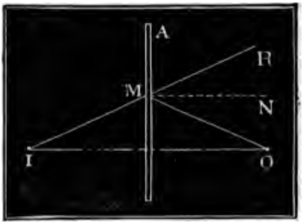

Fig. 4.

similar demonstration might be applied to any other reflected ray; hence all the rays after reflection proceed as if they came from the point I. I is therefore the image of O.

Similarly, every other point of the object has its image as far behind the mirror as the point is in front. The images of all the points in the body form the image of the body.

To draw the paths of the rays by which the image is seen.

12. Of all the reflected rays appearing to proceed from one point of the image only a small pencil will enter the eye of the observer. In order to determine this pencil we must first determine the image I of the point O by the preceding law; then draw lines from the point I to the eye of the observer. Let A B be the points of the mirror in which the extreme line intersects the mirror; then the pencil of rays which comes

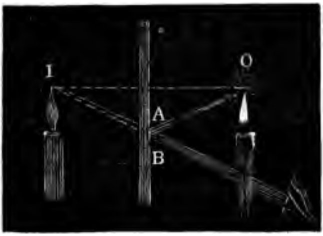

Fig 5.

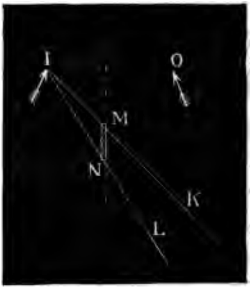

Fig. 6.

originally from O will be reflected at A B, and reach the eye of the observer E. A similar pencil may be drawn for every point of the object. It will be seen that only the part A B of the mirror is concerned in the reflection of the light to the eye.

Limits of visibility of the image.

Let I K, I L, be straight lines from a point I of the image through the limits M and N of the mirror, then it is evident that the eye of the observer must be within the angle K I L, in order that it may see the image I. (Fig. 6.)

Continued Reflections at Plane Mirrors.

Parallel Mirrors.

13. Let M N, P Q, be sections of two parallel mirrors, bisected by a plane perpendicular, and let O be a point of the object in this plane. O has an image $I_{,}$ to the left of M N, and an image I' to the right of P Q. Light from O, after reflection by M N, proceeds as if it came from $I_{,}$ and some of this light falls on P Q.

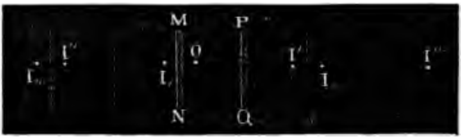

Fig. 7.

This will form an image of $I_{,}$ to the right of P Q; similarly an image I'' of $I_{,}$ will be formed as far to the left of M N as I' is to the right. These images again will have other images, so that an endless succession of images will be formed to the right and to the left of M N, P Q. If we mark off, on each side of the mirrors, distances $a_{\prime}$, $a_{\prime\prime}$, $a_{\prime\prime\prime}$, &c., and a', a'', a''', &c., each equal to the distance between the mirrors, one image will fall in each of these spaces thus marked off.

Images by two Plane Mirrors inclined to one another.

14. Let C A, C B, be perpendicular sections of two plane mirrors. Let Q be the position of an object within the angle A C B formed by the mirrors. With C as centre describe a circle passing through Q, then it will be proved that two series of images will be formed, all of which lie on the circumference of this circle.

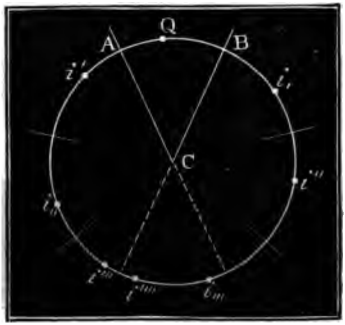

Fig. 8.

Let i' be the first image formed by A C, then since i' is determined by drawing a perpendicular to the *mirror* A C, and measuring an equal distance on the *opposite* side, i' and Q are equidistant from C. Again,

i' has an image i'' formed by the mirror B C, and found by drawing $i'\,i''$ perpendicular to B C, and of such a length that the line $i'\,i''$ is bisected by B C; hence, i' and i'' are equidistant from C. Similarly, Q has an image $i_{,}$ formed by C B, $i_{,}$ an image formed by A C, and so on, and it may be shown that all the images are equidistant from C; hence they all lie on the circumference, having C as centre and C Q as radius.

When we arrive at an image as $i_{,,,}$ or i'''' in the figure falling within the angle formed by continuing A C and B C, that image is the last of the series; for since it falls at the back of both mirrors, no light proceeding from it can fall on the face of either mirror.

Set off from A and B arcs equal to A B, then there will be one image in each arc, excepting in those which overlap the arc $A'\,B'$.

If the angle A C B be an aliquot part of 180°, the images of every point in the arc A Q B will occur the same number of times, and the whole series will be symmetrical. We will investigate the number of images in several cases.

Two mirrors at right angles.

15. Let the mirrors be at right angles (Fig. 9), and let O be the luminous point. There will be one image, I', as far behind A C as O is in front, and another, I''', as far behind B D as I is in front. There will be an image of I' as far behind the continuation of B D as I is in front, and an image of I''' as far behind the continuation of C A as I''' is in front, but these last images will coincide at the point I''; hence there are three images.

To draw the direction of the rays by which the several images will be seen.

16. Let E mark the position of the eye. Draw the line E I′, cutting A C in C, and also draw O C, then the ray by which I′ is seen proceeds from I to C and from C to E. Now draw the line E I‴ intersecting B D in D, and also draw D O. The light by which I‴ is seen proceeds from O to D and from D to E.

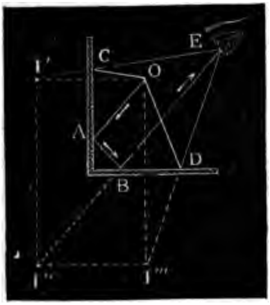

Fig. 9.

Again draw the line I″E. This line will either intersect A C (in which case I″ will be seen as the image of I‴) or B D (in which case I″ will be seen as the image of I′). Suppose E I″ to intersect B D in B, draw B I′ intersecting A C in A, and draw A O. The light by which I″ is seen proceeds from O, is reflected at A along A B, and again at B along B E.

Images of an object formed by inclined plane mirrors.

17. What has been said of the image of a single point applies equally to all points of an object. Suppose, for example, that the mirrors are at right angles, there will be three images of the object (Fig. 10).

Again, let the magnitude of the angle be 60°. At the point O (Fig. 11) we may construct six angles of

60° each; in one of these, namely, the angle A B C, there will be the object, and in each of the other five there will be an image of this object.

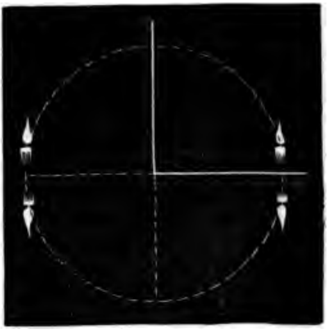

Fig. 10.

If the angle be 45°, we shall have eight angles, and therefore seven images.

The laws of the formation of images with inclined mirrors may be thus collected:—

1st. If the angle of the mirror be contained an exact number of times, as, for instance, n times in 360°, the number of images is in general n; but the number is reduced to $n - 1$ in two cases, (a) when n is even, (b) when the object is on the bisector of the angle between the mirrors.

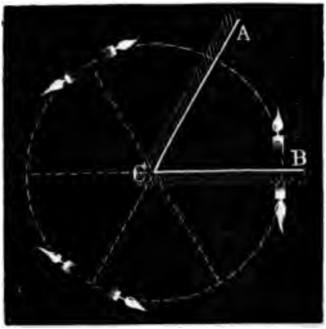

Fig. 11.

2nd. If the angle of the mirror be contained n times in 360° with a remainder, the number of images is at least n, and for certain positions of the object the number is $n + 1$.

EXAMINATION.

1. Define the terms Pencil, Ray, Focus of Light; and also the terms Angle of Incidence, Angle of Reflection, Angle of Refraction.

2. State the Laws of Reflection, and describe any experiment by which they may be illustrated.

3. Explain generally the mode of the formation of images.

4. Prove that the image of a luminous point placed before a plane mirror is on the continuation of the perpendicular to the mirror through the point, and is as far at the back of the mirror as the point is in front of it.

5. Show that in a plane mirror the image is of the same dimensions as the object.

6. How is it that, when the image of a man who is writing is seen in a looking-glass, the man appears to write with his left hand, and the writing seems to slope to the left instead of the right?

7. The image of a candle is visible in a looking-glass, draw the paths of the rays which enable an observer to see it.

8. Show how to determine the limits of visibility of the image of an object in a room, on the wall of which is a small plane mirror.

9. Two large plane mirrors are fixed to the opposite walls of a square room, examine the number and positions of the images of an object placed between them.

10. A ray of light from a point A is reflected at a plane mirror B, and proceeds to a point C according to the Law of Reflection; show that the course A B + B C is the shortest possible course between A, the mirror, and B.

11. Show how to determine, by a geometrical construction, the number of images formed by two plane mirrors inclined at a given angle.

12. On two adjacent walls of a square room there are two

large plane mirrors, how many images of a candle in the room may a person see?

13. Show how to draw the course of the rays by which the images in the preceding example are seen.

14. How many images would the person see of himself?

Spherical Mirrors.

18. Spherical mirrors are of two kinds: *concave*, as the interior surface of a watch-glass; and *convex*, as the exterior. A section by a plane passing through the centre is termed a principal section, and the middle point of the arc of this section is termed the apex. The straight line through the centre and apex is the principal axis.

To find the position of the principal focus of a concave spherical mirror.

19. Let SR be the path of a ray of light parallel to the axis and incident at R on a concave mirror, of which A is the apex and C the centre. The normal at R is the radius CR; hence, if RF be the direction of the ray after reflection by the law of reflection, $\angle SRC =$

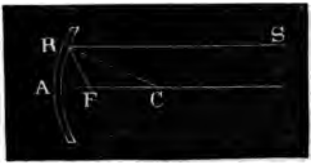

Fig. 12.

∠FRC. Therefore also ∠FCR = ∠FRC, and consequently FR = FC. Now let us suppose R very near to A, then FR = FA nearly, and F bisects AC. This is true for all rays parallel to the axis and near to it; hence every ray of a small pencil of parallel rays, the axis of which is the axis of the mirror, will, after reflection on a concave spherical mirror, meet the axis at a point equally distant from the mirror and its centre.

Principal Focus.

20. The point *f*, to which every ray of a small pencil parallel to the axis converges, is termed the principal focus of the mirror. The distance of the principal focus from the mirror is termed the *Focal Length* of the mirror; hence, the focal length of a spherical mirror is equal to half the radius.

To find, by a geometrical construction, the image of a luminous point.

21. Let C be the centre of the mirror, and let P be the luminous point. To determine the position of the image of P, it will be sufficient to find the point of intersection of any two rays after reflection. Let one of the rays be that which passes through the centre, and will therefore, after reflection, return along the same line. Let PR be any other ray, let CB be a line through the centre parallel to PR, and let the point *f* bisect CB. Then, by regarding CB

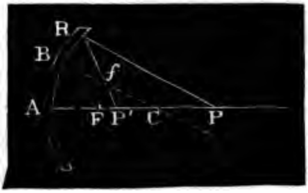

Fig. 13.

as a principal axis, *f* its principal focus, and P R a ray parallel to it, we see that after reflection the ray will pass through *f*. Hence, if R *f* be continued to meet P C in P′, then P′ will be the image of P.

By repeating this construction for different positions of P, it will be seen that—

1st. When the point is at a very great distance from the mirror its image will be at the principal focus.

2nd. As the point approaches the mirror, the image moves from F towards C.

3rd. When the point is at the centre, all the rays being normal to the surface are reflected along the same lines; hence, in this case, the point is its own image.

4th. When the point passes the centre to the left its image moves to the right; for example, when the point is at P′ the image is at P. Hence, each of the points P and P′ is the image of the other. On this account they are termed *conjugate* foci.

5th. When the point is at F, the reflected rays are parallel to one another and to the axis.

6th. When the point P has passed F and lies between

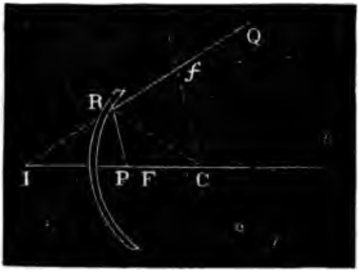

Fig. 14.

F and A (Fig. 14), RQ, the reflected ray, will never meet the axis. If, however, we continue QR, and also continue the axis, beyond the mirror they will intersect in a point I; hence, an observer situated within the pencil of reflected rays, will receive the light from the point P as if it came from I. Hence, I is said to be the virtual focus of P.

22. It is evident, therefore, that every point on the axis, excepting the principal focus, has an image on the same axis.

Real and virtual foci.

23. A focus is the point of intersection of the directions of all the rays of a pencil after incidence on a reflecting or refracting surface. Let us suppose the pencil to traverse a dark room. There are always particles of dust suspended in the air which scatter the light and make the pencil visible; hence, the point upon which the rays are concentrated will appear as the common apex of the two cones formed, one by rays converging to the point, and the other by rays which having passed the point diverge from it. If a screen of paper be placed so as to intercept the light at the point, a bright spot will be seen on the screen. Hence, the focus is said to be *real.*

When the rays of light actually pass through the focus, it is said to be real. When the rays themselves do not meet in the focus, but only the continuations of their lines of direction, the focus is said to be virtual. A virtual focus cannot be received on a screen; it is only an appearance. Rays of light may be received by the eye *as if* they come from the virtual focus, but this is not really the case. A real focus may be received on a screen, and when seen by the eye the rays really come from it, having passed through it after reflection or refraction.

Images of objects formed by concave mirrors.

24. When an object is placed before a concave mirror, every point in the object has its image. The collection of these points form the image of the object. The image is real when the light passes through it, and virtual when the light does not pass through it. The following diagrams will illustrate this difference. In Fig. 15 the candle is placed in front of the concave mirror, and at a distance from it greater than the focal

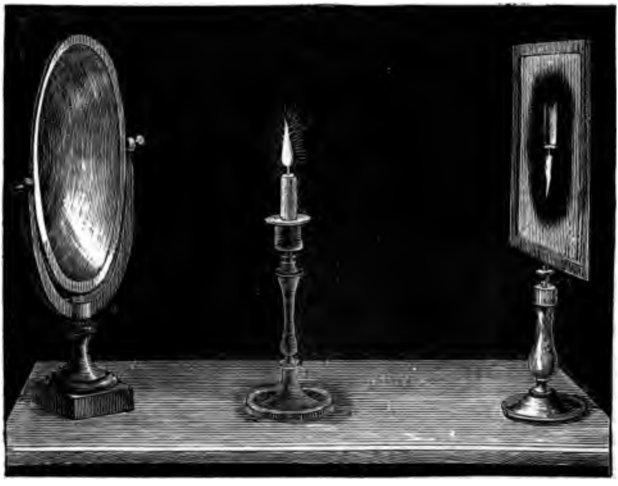

Fig. 15.

length but less than the radius. If a screen of paper be placed near the candle and gradually moved away, a position will be found in which a distinct inverted image of the candle will appear on the screen. This is a real image, and will be greater than the object, for always *the size of the image is to the size of the object as their distances from the mirror.*

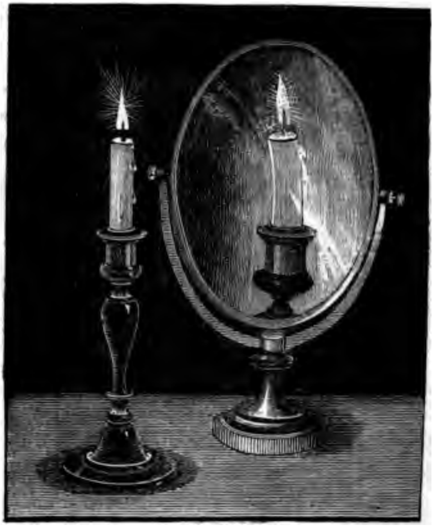

Fig. 16.

If the candle be placed, as in Fig. 16, nearer to the mirror than the principal focus, the image cannot be received on the screen; but, on looking at the mirror, the image may be seen apparently behind it, erect and *magnified.* The image in this case is virtual.

Convex Mirrors.

To find the principal focus.

25. Let MAL be a perpendicular section of the mirror, C its centre, and CA its axis. Let SR be a ray parallel to the axis and incident at R; let RT be the direction after reflection at R; and also let CN be the normal at R. Then by the law of reflection ∠NRT = ∠NRS, but ∠NRT = ∠CRF and ∠NRS = ∠RCF, therefore ∠CRF = ∠RCF, and consequently CF = RF. But when the pencil is very small, R is very near to A, and FR is nearly equal to FA; hence we may say that when the pencil is very small CF = AF. The focal length of the mirror is therefore equal to half the radius, and being at the back of the mirror is *negative*.

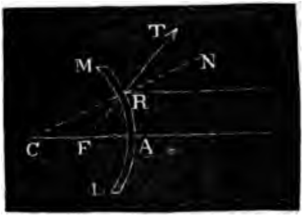

Fig. 17.

To find the image of a luminous point on the axis.

26. Let C be the centre and A the vertex of the mirror (Fig. 18), and let P be the luminous point. The image of P may be determined by finding the point of intersection of any two rays from P after re-

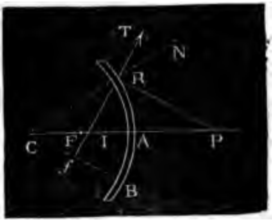

Fig. 18.

flection. Let one of these be PA, and let PR be any other ray. The refracted ray RT will not meet AP, but if both these lines be produced beyond the mirror they will meet. To determine the direction of RT, draw CB parallel to PR and bisect CB in *f*. Then, by regarding CB as the principal axis, to which the ray PR is parallel, we see that after reflection the ray RT will have the direction *f*R. Consequently the point I, in which *f*R cuts AC, is the image of P.

It is evident, therefore, that the image *always* lies between F and A when the mirror is convex, and is therefore always virtual.

From §§ 21, 25, and 26, it will be seen that the image of any point P is always on the line passing through the point P and the centre of the mirror.

To find by a geometrical construction the position and size of the image of a small object placed before a spherical mirror.

27. Let C be the centre and F the principal focus of the mirror, and let MN be the object. By § 22 the

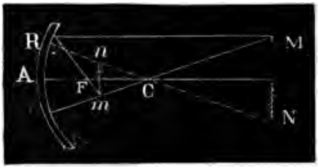

Fig. 19.

image of the point M will lie in the line MC. Again, let MR be the ray from M parallel to AC; by § 19, this ray after reflection will pass through F. Let RF meet MC in *m*, then *m* is the image of M. Similarly, we may find the image of N by first drawing N C, then a line N Q parallel to A C, and, thirdly, drawing the line Q F to meet N C in *n*. Hence *mn* is the image of the object M N. The same construction applies to a convex mirror. It will be at once seen that if the image and object are on the same side of the centre, the image is erect; but if they are on opposite sides of the centre, the image is inverted. The size of the image is to that of the object as their distances from the centre C, and also as their distances from A.

To show the course of the rays by which the image is seen.

28. Let M N be the object. First find the image by § 27. Let *mn* be the image. Next mark the position, E, of the eye. From E draw lines through *n*, and from the points in which they meet the mirror draw lines to N. These lines represent the pencil of rays by which the point N is seen. To the eye the rays seem to proceed from *n*. Simliarly, by drawing lines from

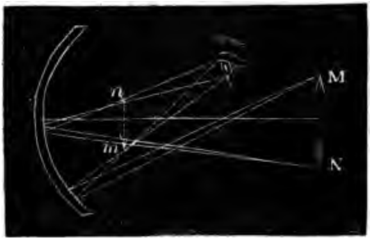

Fig. 20.

the eye through *m*, and continuing them from the points in which they meet the mirror to M, we obtain the pencil by which M is seen. Pencils from intermediate points will lie between these extreme pencils.

The formulæ connecting the position of a luminous point with that of its image.

29. Let u be the distance of the point from A, and v the distance of its image from A; let f be the focal length, and r the radius; then the following formulæ are always true:—

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}=\frac{2}{r}\text{ for a concave mirror.}$$

$$\frac{1}{v}+\frac{1}{u}=-\frac{1}{f}=-\frac{2}{r}\text{ for a convex mirror.}$$

The formula may be proved by referring to Fig. 13. In the similar triangles $P'Cf$ and $P'PR$

$$\frac{CP'}{Cf}=\frac{PP'}{PR}$$

but $CP'=r-v$, $Cf=\frac{1}{2}r$, $PP'=u-r$, and when the pencil is small $PR=PA=u$ nearly.

$$\therefore\quad\frac{r-v}{\frac{1}{2}r}=\frac{u-v}{u}$$

$$\therefore\quad\frac{1}{v}+\frac{1}{u}=\frac{2}{r}$$

EXAMINATION.

1. Define the principal focus of a mirror, and find the position of the principal focus, (*a*) for a concave spherical mirror, (*b*) for a convex spherical mirror.

2. Show how to find by a geometrical construction the position of an image of a point in the axis.

3. Show how to find the position of the image of a point *not* on the principal axis through the vertex of the mirror.

4. Distinguish between real and virtual images.

5. Show how to find by a geometrical construction the position and magnitude of an image of an object placed before a concave mirror.

6. Draw the course of the rays by which an observer in a given position would see the image.

7. Explain the meaning of the formula

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

8. Write this formula in the form, (*a*) for a concave mirror, (*b*) for a convex mirror.

9. A concave mirror, the focal length of which is 12 inches, is held at a distance of 18 inches in front of a candle; where will the image be? Can the image be received on a screen?

10. The image of an object is distant 8 inches from a convex lens of focal length 12 inches; find the distance of the object.

11. What is the focal length of the convex mirror which will produce at a distance of 12 feet the virtual image of an object which is 3 feet distant from the mirror?

12. From the formula of Question 7 prove that with a concave mirror, (*a*) the image is real when u is greater than f, (*b*) the image is virtual when u is less than f.

13. Prove from the formula that with a convex mirror the image is always virtual.

REFRACTION.

30. Although a ray of light will always move in the same straight line when it is not interrupted, yet, when light passes through glass or a transparent fluid, it does not reach the eye or illuminate a screen in the same manner as before it was intercepted by the trans-

parent body. This obviously arises from the direction of the light being changed by some power which resides in the body. In order to understand how this change is produced, let M N O P be a vessel, in one of the sides of which, N O, there is a small hole at Q. If we place a lighted candle within two or three feet of the hole, so that its flame may be at A, a ray of light A Q proceeding from it will pass through the hole Q, and go on in a straight line, Q C, till it reaches the bottom of the vessel at C, where it will form a small circle of light. Having put a mark at the point C, pour water into the vessel till it rises to the height *s t*, and you will see that the round spot which was formerly at C is now at D, that is, the ray of light A Q, which when the vessel was empty went straight on to C, has been bent at the point B, where it falls on the water into the line B D. If we mix a little soap with the water, so as to give it a slight turbidity, the path of the ray B D will be distinctly visible, and it will be seen that it is a straight line, and that the bending or change in its direction has been produced wholly at the point B in the surface of the water. This bending of the ray is called refraction.

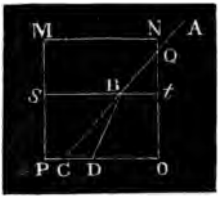

Fig. 21.

31. If, instead of fresh water, we pour in salt water, it will be found that the ray *a b* is more bent at B. In like manner, alcohol will refract the ray more than salt water, oil more than alcohol, and glass more than oil.

Hence we may conclude that whenever a ray of light passing through air falls obliquely upon the surfaces of transparent liquid, or solid bodies, it is refracted by them, and by different bodies in different degrees.

32. If, when the M N O P is empty, we fix C, a bright object, such as a sixpence, and place the eye at A, so that C Q and A are in the same straight line, the sixpence will be distinctly seen by rays which proceed from C, and pass through the aperture to A. Let water be now poured into the vessel to the height *s t*, the eye at A will no longer see the sixpence at C; but if we move it from C to D, it will become visible.

33. The same effect will be produced if *s t* is the surface of salt water, alcohol, oil, or glass; but with these substances we must push the coin beyond D towards O, in order that it may be seen from A. Hence we may conclude that when a ray of light passes through a liquid or solid body in a direction oblique to the surface on leaving the body, the ray is refracted, the amount of refraction being different for different bodies. The following law is proved by experiment to hold good in all these cases.

Laws of Refraction.

34. 1st. The incident ray, the refracted ray, and the normal, at the point of incidence, lie in the same plane.

2nd. The ratio of the sine of the angle of incidence to the sine of the angle of refraction is always the same for the same media.

Explanation of the term Sine.

35. If a perpendicular be let fall from one of the sides of an angle to the other side, the ratio of this perpendicular to the hypothenuse of the right-angled triangle formed is termed the sine of the angle. Thus the sine of the angle B A C is the ratio $\frac{BC}{AB}$.

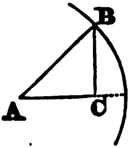

Fig. 22.

Suppose the length, A B, to be taken equal to a unit of length,—that is to say, let A B = 1, then the length of the perpendicular exactly represents the value of the sine. For example, let A B be taken equal to 1 foot, and suppose that B C is then found to be ·58 feet, the sine of the angle B A C is ·58.

36. Let L O be a ray incident at O (Fig. 24) on a refracting surface, and refracted along O R; let D O be the normal at O. By the first law of refraction, L O, O R, and D O are all in the same plane. Let a circle be described in this plane, having O for centre, and meeting the directions L O, O R, in L and R respectively. Draw L S and R T perpendicular to O C, then whatever the angle of incidence may be, the ratio $\frac{LS}{RT}$ remains the same.

The ratio of the sine of the angle of incidence to the sine of the angle of refraction is termed the *refractive index* for the given media.

37. *Having given the direction of the incident ray and the refractive index, to determine the direction of the refracted ray.* Let A O be the incident ray, let a circle with O, as centre in a plane perpendicular to the surface, intersect O A in A, and let A C be the perpendicular to the surface. Divide O C by the refractive

index, and set off the quotient O D on the other side of O. Draw D B perpendicular to the surface, and meeting the circumference in B, then O B is the direction of the refracted ray, for it will be seen that $\frac{OC}{OD}$ is the same as the ratio of the sines, and this ratio is by construction equal to the refractive index. Hence O B is the ray required.

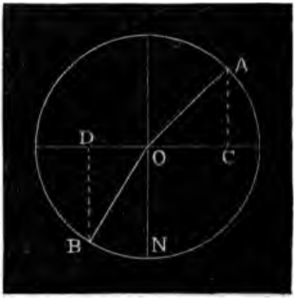

Fig. 23.

38. When light passes from a rarer into a denser medium, the value of m is greater than unity, and consequently the angle of refraction is less than the angle of incidence. When light passes from a denser into a rarer medium, m is less than unity, and the angle of refraction greater than the angle of incidence. Hence, by refraction, a ray of light is bent towards the perpendicular when passing into a denser medium, from the perpendicular when passing into a rarer medium.

Between atmospheric air and plate or crown glass, $m = \frac{3}{2}$ nearly; between atmospheric air and water $m = \frac{4}{3}$. When the light passes from glass or water into air, the refractive indices are the reciprocals of the values just given, and are therefore $\frac{2}{3}$ and $\frac{3}{4}$ respectively. For bisulphide of carbon into air $m = {\cdot}59$. Alcohol to air $m = {\cdot}71$.

Experimental Proof of the Laws of Refraction.

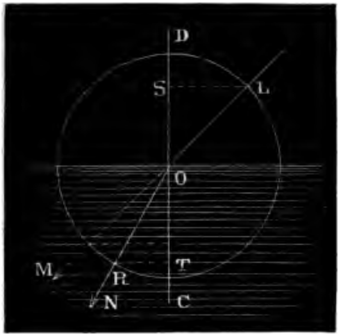

Fig. 24.

39. Descartes, to whom is due the discovery of this law, employed a spherical glass vessel exactly half full of water, and caused a small pencil of light to enter the water at the centre. He then found that if the pencil passed into sphere at L (Fig. 24), and out at R, the ratio of L S, the distance of L from the vertical through the centre to R T, the distance of R from the same vertical was constant.

The law is best illustrated experimentally by means of a vessel formed of a ring of glass and two circular glass plates. The vessel is filled with water up to the centre. At the back of the glass is a rod L M (Fig. 24), movable about the centre O, so that it can be made to take the direction of the incident ray, and a rod O N also movable about the centre, to indicate the direction of the refracted ray. The length of the rod, O M, is exactly equal to that of O N. A single beam of light is made to enter the vessel at a point, L, and to proceed to O, and the rod L M is moved to take the *direction* of the beam. A screen is then placed below the vessel, and the rod O N is made to take the direction of the refracted ray, by causing it to point to the spot which is illuminated on the screen. A movable scale then gives the distances of M and N, from the vertical through O. It is found that whatever may be the angle made by the rod L M, the ratio of these distances is always the same.

The limiting angle of refraction.

40. Let m be the refractive index for two given media, the light passing from the rarer to the denser, and let A O (Fig. 23) be a ray incident obliquely at O. By the construction of § 37, draw O B, the refracted ray. Then it has been shown that $\frac{\text{O C}}{\text{O D}} = m$, or $\text{O D} = \frac{\text{O C}}{m}$. Now let A O be inclined more and more to the vertical, so that it approaches the surface; then O C increases, and becomes more and more nearly equal to the radius of the circle, and then O D approaches the radius ÷ m. Thus, the limit of O D is $\frac{\text{O B}}{m}$, and the limit of the angle B O N is such as will give $\frac{\text{D O}}{\text{B O}} = m$. Let us denote this angle by θ. If the light proceed from B along B O, then it will emerge along O A, provided the angle B O N be less than the limit θ. If the angle B O N be greater than θ, the light will not emerge, but will be turned back at the surface according to the laws of reflection. The ratio $\frac{\text{D O}}{\text{B O}}$ is equal to the sine of the

angle B O N (§ 35); consequently, if m be the refractive index for the rarer to the denser of two media, the angle whose sine is equal to $\frac{1}{m}$ is termed *the limiting angle of refraction*, or *the limiting angle of total reflection.* It is sometimes also termed the *critical angle.*

For instance, suppose the refractive index for air to water to be $\frac{4}{3}$. Construct the right angle D O N with a radius 4, and with centre, O, describe a circle, cut off O D = 3, and draw D B perpendicular to O D (Fig. 23). The angle B O N will be the limiting angle of refraction. No light which passes through water to the surface, and is incident at an angle greater than B O N, can pass out of the water. It will be totally reflected.

41. Let O be a luminous point under water (Fig. 25); by the above construction find the limiting angle θ

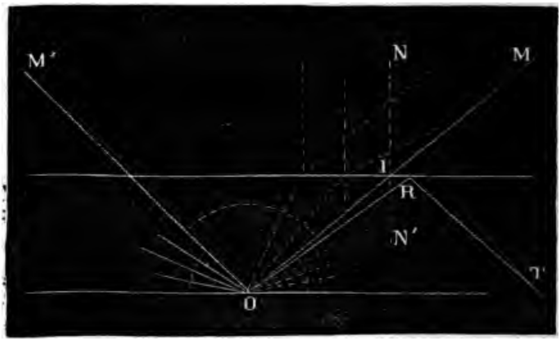

Fig. 25.

(about 48·5°), and through O draw O M, O M′, each making an angle θ with the vertical through O; then rays of light passing from O to the surface within the angle M O M′ will emerge, but any ray O R falling without the angle will be totally reflected in the direction R T. The effect both of refraction and of total reflection may be easily exhibited by taking a glass tumbler of water with a spoon in it, and holding it above the eye (Fig. 26), so as to see the surface of the water from below. The image of the spoon will be seen as in a mirror, by reflection. The part in the water will be seen by rays which are refracted so that

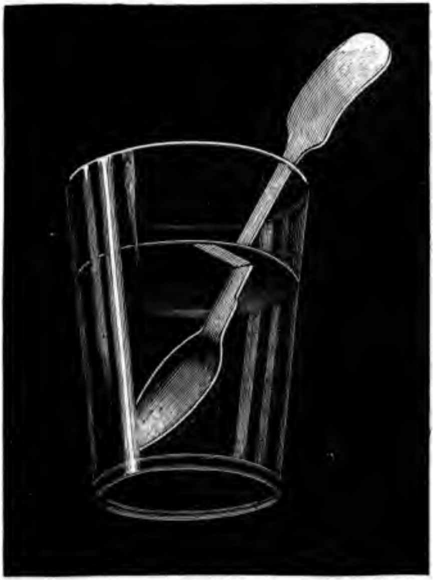

Fig. 26.

it will not appear as the direct continuation of the part without the water.

Effects of Refraction.

To find the relation between the true and apparent depths of an object under water.

42. Let L be a luminous point in a dense medium, as water. Let L A and L B be two rays from L, and let A C, B D, be their directions after passing into the air, and let the continuations of C A and C B meet in L′; then the eye which receives the pencil of which L A and L B are the extreme rays, sees the point L as if it were at L′, hence L′ is the image of L. To find the position of L′ when the pencil is vertical, let us consider only the ray L A C. Let K L be the perpendicular to the surface, and let it meet the surface in N. When L A is very near to L K, L′ is on L K.

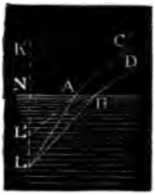

Fig. 27.

$\frac{NA}{AL'}$ is the sine of the angle between C A and the normal;

$\frac{NA}{AL}$ is the sine of the angle between L A and the normal.

But by the law of refraction

$\frac{NA}{AL'} \div \frac{NA}{AL} = m$, the refractive index for air *into* water;

therefore $AL = m \cdot AL'$.

If A be very near N then $NL = m \cdot NL'$.

Thus, if d be the true depth of a stream, and d' its apparent depth,

$$\text{then } d = d' \cdot m.$$

Since m is greater than 1, an object in water will not seem to be so deep as it really is. For example, a fish under water will seem to be nearer the surface than it is in reality. A sportsman aiming at the fish as he sees it will fire above it; he must aim lower than the image of the fish in order to hit it.

43. If a straight stick be placed in water, the image of the immersed part will be above the part itself (§ 42), consequently the stick will appear bent, as in the figure.

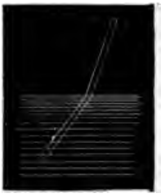

Fig. 28.

Passage of a ray through a transparent plate with parallel faces.

44. When a ray passes through a transparent plate with parallel faces, it emerges in a direction parallel to the direction of incidence. Let M N be the plate, S A the incident ray, A B the direction after refraction at the first surface of the glass. Because the surfaces are parallel, the angle K B A, which is the angle of incidence at the second

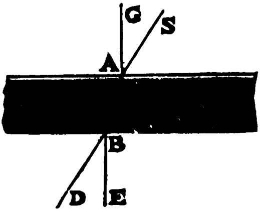

Fig. 29.

surface, is equal to the angle I A B: and since if the light were reversed it would proceed along the same path, the angle after emergence E B D = S A G, and therefore A S and B D are parallel.

In the same way it may be shown that if the ray passes through any number of plates with parallel faces, some of which are repeated, its directions in the same medium are always parallel. For example, if a ray pass through a series of plates as follows, the parts in 1 and 4 will be parallel, and so will be the parts in 2 and 5.

1 *water*
2 *glass*
3 *air*
4 *water*
5 *glass*

Deviation of a ray in passing through a prism.

45. A medium bounded by two faces which intersect is termed a prism (Fig. 31). A plane perpendicular to the two intersecting faces is termed a principal plane.

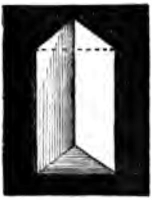
Fig. 30.

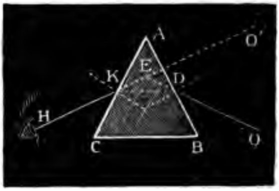

Fig. 31.

Let O D be a ray from a luminous point O in the principal plane, which cuts the prism in the section A B C. The ray is bent towards the normal at D, and on emergence it is bent again away from the normal at K. The ray is received by the eye H as if it came from O′. The angle between the emergent ray K H and the incident ray O D is the angle of deviation. It is found that *the deviation is always towards the thickest part of the prism.* The deviation is a minimum, that is to say, it is the least possible when the angle of incidence and the angle of emergence are equal (Fig. 31).

LENSES.

46. Any portion of a refracting medium which has two opposite surfaces, either both spherical, or one spherical and the other plane, is termed a *lens.*

47. The combination of spherical surfaces, either with each other or with plane surfaces, gives rise to six kinds of lenses, sections of which are represented in Fig. 33; four are formed by two spherical surfaces, and two by a plane and a spherical surface.

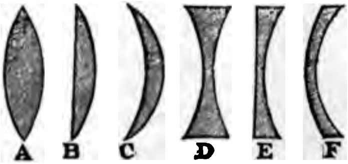

Fig. 32.

A is a *double convex*, B is a *plano-convex*, C is a *converging concavo-convex*, D is a *double concave*, E is a *plano-concave*, and F is a *diverging concavo-convex*. The lens C is also called the *converging meniscus*, and the lens F the *diverging meniscus*.

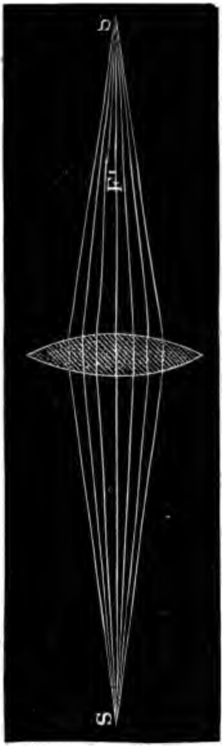

Fig. 33.

48. From § 45 we may at once conclude that when a ray passes through a lens the deviation is always towards the thickest part (Fig. 33). Hence a pencil of light may be made *converging* by a lens which is thickest in the middle, but not by a lens which is thinnest in the middle. Consequently the first three lenses, which are thicker at the centre than at the borders, are termed *converging*; the others, which are thinner in the centre, are *diverging*. In the first group, the double convex lens only need be considered; and in the second, the double concave, as the properties of each of these lenses apply to all those of the same group.

49. The straight line which passes through the centres of the two spherical surfaces of the lens is termed the *principal axis*.

50. In every lens there is a point such that for every ray which passes through the point the direction after emergence is parallel to the direction before incidence. This point is termed the *optical centre* of the lens. When the thickness of the lens is small, as is usually the case, the distance between the two parallel parts of the ray is also small. We may, therefore, without sensible error, suppose a ray passing through the optical centre to proceed without refraction, that is to say, in the same straight line.

Converging Lenses.

The principal focus.

51. When a small pencil of rays parallel to the axis passes through a double convex lens, it converges to a point *f* (Fig. 34), which is termed the *principal focus* of the lens. The distance *cf* of the principal focus from the lens is termed the *focal length* of the lens. The focal length is always the same for the same lens, but it varies with the radii of the two surfaces and with the refractive index. The focal length may be expressed in terms of these quantities, but it will suffice

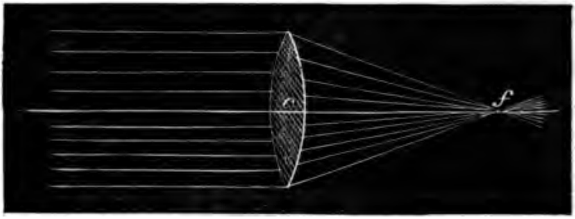

Fig. 34.

for our present purpose if we suppose the focal length always to be determined experimentally. Since rays may be incident on either side, it is evident that there are two principal foci, one on each side of the lens, and (neglecting the thickness of the lens) equidistant from the optical centre.

If rays from a luminous point at a great distance pass through the lens, they will meet at the principal focus. Hence we may determine the focal length of a convex lens by forming with it a distinct image of the sun on a sheet of paper, and then measuring the distance of the image from the lens.

If rays from a luminous point placed at the principal focus pass through the lens, they will emerge parallel and will not converge to form an image of the point.

It is evident, therefore, that any ray whatever which passes through the principal focus will after refraction be parallel to the principal axis.

To find the focus of a small pencil of rays parallel to one another, but not to the principal axis.

52. Let C be the optical centre, and F or F′ the principal focus of the lens. To determine the focus, it will

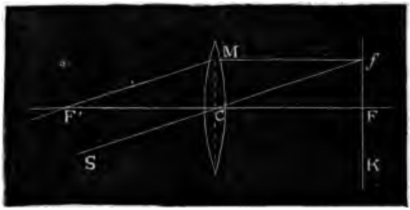

Fig. 35.

be sufficient to find the point of intersection of any two rays. The most convenient rays for the purpose are the ray S C, which passes through the centre, and which we may suppose to continue in the same straight line, and the ray F′ M, which passes through the principal focus F′, and consequently after refraction is parallel to the principal axis. Let *f* be the point of intersection of these rays, then *f* is the focus of the pencil. Since *f*M F′C is a parallelogram, M*f* = CF′ = CF, therefore *f* is on the line K F*f* drawn through the principal focus *f* perpendicular to the axis.

Any other ray parallel to S C will after refraction pass through *f*; hence the above proposition affords an easy way of determining the direction of any ray P R after refraction (Fig. 36). The rule will be—draw a line S C through C parallel to P R, and let it meet F K in *f*, then R*f* is the direction of the refracted ray.

To find the image of a luminous point on the principal axis.

53. Let C be the optical centre of the lens, and let F be the principal focus on one side, and F′ the principal focus on the other side. Also let F′K be a perpendicular to the axis at F′.

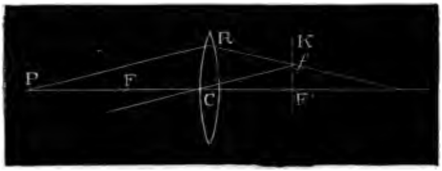

Fig. 36.

1st. Let P be further from the lens than F. The image of P may be determined by finding the point o. intersection of any two rays from P which pass through the lens. One of these rays may be P C, or that which passes through the centre. Let P R be any other ray. Through the centre C draw M C *f* parallel to P R, and let it meet F′K in *f*. Then by § 52 P R passes through *f*, and the image of P will be found by producing the line R*f* to meet the axis in I. Now the triangles P R C, C*f*F′, have these sides respectively parallel, and consequently proportional; hence, as C F′ is by hypothesis less than P C, therefore *f*F is less than C R, and the point I lies beyond F′. It is evident, therefore, that when the point P is beyond the principal focus on one side, its image is beyond the principal focus on the other side. The image is therefore real.

54. If P be distant twice the focal length from the lens, that is, if C P = 2 C F, it may be easily demonstrated by means of the above construction that C I = 2 C F′; hence, in this position, the conjugate foci P and I are equidistant from the lens.

2nd. Let P be nearer to the lens than the principal focus. As in the previous case, let P R be any ray, let

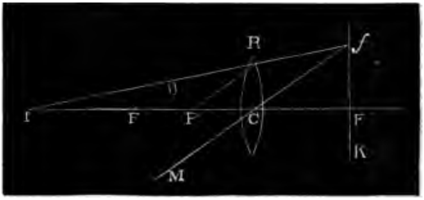

Fig. 37.

M C, drawn through the centre C parallel to P R, meet F′K in the point f. Then Rf is the direction of the ray P R after refraction (§ 52). Let fR be continued to meet the axis in I. As the sides of the triangles R P C, fC F′, are parallel, they are also proportional; but as C F′ is by hypothesis greater than C P, therefore F′f is greater than C R, and I lies beyond P on the same side of the lens. As the light from P does not really pass through I, it is a virtual image of P.

These results may, therefore, be summed up thus:—With a converging lens, when a luminous point is at an infinite distance, its image is on the opposite side of the lens at the principal focus. As the luminous point moves nearer to the lens, its image moves further away. When the point is at a distance equal to twice the focal length, the image is also at a distance equal to twice the focal length on the opposite side of the lens. (Fig. 33.) When the point reaches the principal focus on one side, the image has passed to an infinite distance on the other. Finally, when the point is nearer the lens than the principal focus, the image is further from the lens than the point, but on the same side.

To find by a geometrical construction the image of a point not on the principal axis.

55. Let C be the optical centre and F the principal focus of the lens, and let P be the luminous point.

To determine the position of the image of P, it will be sufficient to find the point of intersection of any two rays which proceed from P and pass through the lens. The *two* most convenient rays for the purpose are, the ray passing through the centre and the ray parallel to the axis. Now, the ray P C from P through the centre will proceed in the straight line P C (§ 50), and the ray P E, which proceeds from P parallel to the principal axis, will after refraction pass through F (§ 51). Hence, if we draw P E parallel to the axis, and then draw the line E F meeting the continuation of P C in I, then I is the image of P.

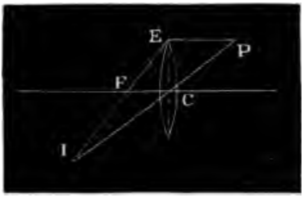

Fig. 38.

Image of an object with a Converging Lens.

56. Let O be the optical centre, and F the principal focus of the lens, and also let A B be the object.

1st. Let A B be at a very great or infinite distance. All the rays from A B to the lens will be sensibly parallel, and will therefore converge to the principal focus. The image will therefore be reduced to a point.

2nd. Let A B be at a definite distance from the lens greater than the focal length. The image of A B may

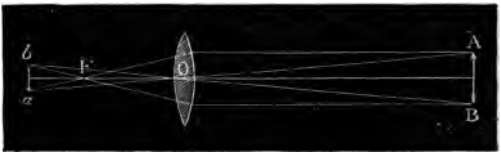

Fig. 39.

be constructed as follows:—Draw from A the ray which passes through the centre O; and also draw the ray A E to the lens parallel to the axis. The first of these

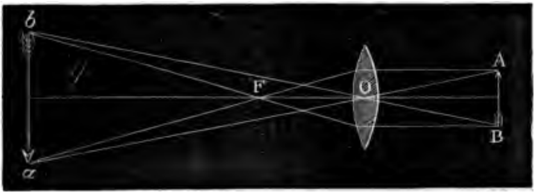

Fig. 40.

rays will proceed in the straight line A O, and the second will pass through the principal focus F; hence, if the line E F be drawn and continued to meet A O in *a*, then *a* is the image of A. Similarly, by drawing B O through the centre, B G parallel to the axis, and G F through the focus, meeting B O in *b*, we obtain *b*, the image of B; *a b* is therefore the image of A B. It is evident that the image is real and inverted, and smaller than the object, when the latter is at a distance greater than twice the focal length, and larger than the object when the latter is at a distance from the lens between once and twice the focal length.

3rd. When the object is at the principal focus, the image is at an infinite distance.

4th. Let A B be nearer the lens than the principal focus *f*. The image of A is obtained, as in the previous case, by drawing A O through the centre A E parallel to the axis, and E F through the principal focus. In this case, however, it will be found that F E and O A do not meet on the side of the lens opposite to A, but at the point *a*, on the same side as A. In the same way we

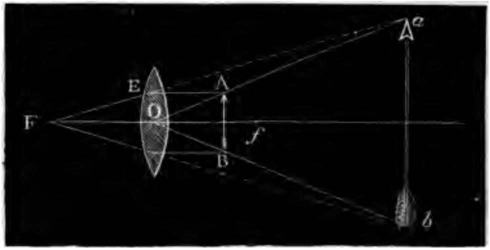

Fig. 41.

may find *b* the image of B. Hence we see that the image is virtual, erect, and larger than the object.

Diverging Lenses.

The principal focus.

57. When a small pencil of rays parallel to the axis passes through a double concave lens, the rays diverge after refraction by § 45. They do not therefore meet the axis in their passage, but if their directions be

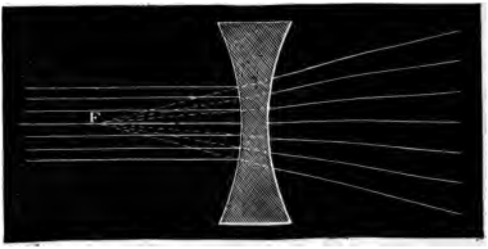

Fig. 42.

continued backwards, these will meet the axis in a point F. This point is the principal focus.

To find the focus of a pencil of rays parallel to one another, but not to the principal axis.

58. Let C be the optical centre, F the principal focus on one side, and F′ that on the other side. Let F K be perpendicular to the principal axis at the point F. To determine the focus of the pencil it will be sufficient to find the point of intersection of any two rays. Let one of the rays be A C, which passes through the optical centre, and let the other be the ray B R, which, if continued in the same straight line, would pass through F′. By § 57, after emergence, this ray B R will have the direction R D parallel to the axis. Continue D R to meet A C in *f*, then *f* is the focus of the pencil. Since *f* R F′ C is a parallelogram, *f* R = C F′ = C F, therefore *f* is on the line F K.

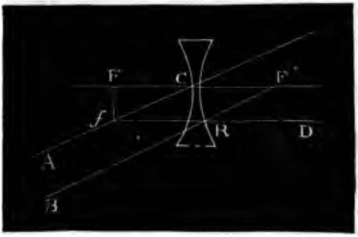

Fig. 43.

To find the image of a point on the axis formed by a diverging lens.

59. Let C be the optical centre, and F the principal focus of the lens. Let F K be perpendicular to the axis

at F, and let P be the luminous point. To determine the image of P it will be sufficient to find the point of intersection of any two rays from P through the lens. Let one of these be PC, which passes through the centre C. Let PR be any other ray, and let a line through C parallel to PR cut FK in *f*, then PR after refraction will have the direction of *f*R (§ 58). Let *f*R cut PC in I, then I is the image of P. It is evident that P and I are both on the same side of the lens, and that I is nearer the lens than P.

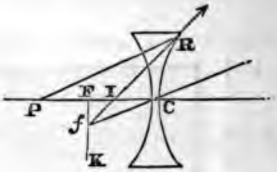

Fig. 44.

Images of objects with a diverging lens.

60. Let [illegible] the cen[illegible]
and F [illegible]
[illegible]
[illegible]

the light after refraction [illegible]
it is always erect and [illegible]

of
sion
bject

is seen
We see, therefore [illegible] measures
convex mirror [illegible] is found
double convex [illegible] 's distance
images when [illegible] of the lens
but virtual images [illegible]
focal length.

most distinct vision
has
= 18.

on the screen. On placing the eye at a distance of about nine inches behind the paper, the image will be distinctly seen on it; and if the paper be taken away while the eye is in a straight line with the image, lens, and object, the image will be still visible at the place at which it appeared on the screen.

The distance nine inches is here mentioned because the eye cannot see objects distinctly at a distance less than nine inches, and for ordinary vision rather more than nine inches is a convenient distance.

The rays proceed to the eye from the image in the same way as from an object having the same position and dimensions. Hence, in order that the image may be seen the eye must be beyond it and at a distance greater than nine inches. For instance, to find the position of the eye for distinct vision in Figs. 39 and 40, we must continue the line O F beyond *a b*, and place the eye at a distance from *a b*, which shall represent the ordinary distance of distinct vision.

If the same lens be placed near any object, as, for example, a page of print, the image cannot be received on a screen, but will be found on the same side as the object (Fig. 41); and, on placing the eye near the lens, the image will be seen erect and magnified. The rays from the object are refracted by the lens, and reach the eye with the same inclination as if they came from an object having the position and dimensions of the image. Hence, if a converging lens having a focal length of about three inches be used to look at a page of print when the lens is held at a distance of a foot *from the* eye the print appears inverted, and when the lens is held close to the eye, and at a distance of an inch from the paper, the print appears erect.

Now, with a diverging lens the image is always virtual; consequently, however near or distant the object may be, it can be seen in an erect position when the lens is held near the eye (Fig. 45).

The magnifying power of a lens.

62. A lens magnifies an object when the image of the object seen at the distance of most distinct vision subtends a larger angle at the eye than the object would subtend at the same distance.

The ratio of the angle under which the image is seen to the angle under which the object is seen measures the magnifying power of the lens. This ratio is found to be nearly equal to the ratio of the observer's distance of distinct vision (d) to the focal length of the lens (f), or

$$\text{Magnifying power} = \frac{d}{f}$$

Thus for a person whose distance of most distinct vision is nine inches, a lens of ½-inch focus has

a magnifying power = 18.

Formulæ showing the relation between the distances of two conjugate foci and the principal focus from the lens.

63. Let u = the distance of a luminous point from the centre of a lens, let v = the distance of the image, and let f = the focal length; then it is found that, if we consider lines measured in a direction opposite to that in which the light comes as positive, and those measured in the same direction as the light as negative,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

f is always negative, for lenses which are thickest in the middle, and positive for lenses which are thinnest in the middle.

This formula may be proved from Fig. 36. From the similar triangles P C R, C F f

$$\frac{\mathrm{CR}}{\mathrm{CP}} = \frac{\mathrm{F}f}{\mathrm{CF}}$$

and from the similar triangles I C R, I F f

$$\frac{\mathrm{CR}}{\mathrm{CI}} = \frac{\mathrm{F}f}{\mathrm{FI}}$$

Hence, if we divide the second equation by the first

$$\frac{\mathrm{CI}}{\mathrm{CP}} = \frac{\mathrm{FI}}{\mathrm{CF}} = \frac{\mathrm{CI}}{\mathrm{CF}} - \frac{\mathrm{CF}}{\mathrm{CF}}$$

$$\frac{1}{\mathrm{CI}} + \frac{1}{\mathrm{CP}} = \frac{1}{\mathrm{CF}}$$

But C P $= u$, C I $= -v$, and C F $= -f$.

Therefore by prefixing the proper signs we have

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

A similar proof applied to Fig. 44 will establish the formula for concave lenses.

Example 1.—A luminous point is 3 inches from a convex lens having a focal length of 5 inches; find the position of the image.

$$\text{Here } f = -5, \quad \text{and } u = 3$$

$$\therefore \frac{1}{v} - \frac{1}{3} = -\frac{1}{5}$$

$$\frac{1}{v} = +\frac{2}{15}$$

$$v = +\ 7\tfrac{1}{2} \text{ inches.}$$

The sign shows that the image is on the same side of the lens as the object.

Example 2.—The image and object are to be six inches apart when the lens is between them; what is the greatest focal length of the convex lens which will give this result?

Here f is negative and v is negative, since it is measured in the opposite direction to that of u,

$$\text{and } u + v = 6.$$

$$\text{Now } \frac{1}{-v} - \frac{1}{u} = -\frac{1}{f}$$

$$\therefore \frac{u+v}{uv} = \frac{1}{f}$$

$$\therefore f = \frac{uv}{6}$$

Hence f is greatest when uv is greatest; but if a line 6 inches long be divided into two parts u and v, the rectangle contained by the parts is greatest when the parts are equal, or $u = v = 3$ inches. When this value is given to u and v,

$$f = \frac{u\,v}{6} = \frac{9}{6} = 1\tfrac{1}{2}.$$

EXAMINATION.

1. State the Laws of Refraction, and describe fully any experiment by which they may be illustrated.

2. Having given the angle of incidence, and the refractive index, show how to determine the direction after refraction.

3. A ray of light falls on the surface of water so as to make with the vertical an angle of 60°. Having given the refractive index $= \frac{4}{3}$, construct the angle of refraction exactly, and give its value in degrees approximately.

4. Why should a lake the bottom of which you can see appear shallower than it really is?

5. What is the relation between the apparent depth and the true depth of an object under water when the observer is vertically above it?

6. What will be the apparent depth of a stream the true depth being 20 feet?

7. Having given the refractive index for air to water $= \frac{4}{3}$, and for air to glass $= \frac{3}{2}$, find the refractive index for water into glass. (Apply § 44.)—*Ans.* $\frac{9}{8}$.

8. If a person below the surface of a lake was able to see an object on the bank, what would be the appearance of the object?

9. How is it that a coin placed in a tumbler looks larger when the tumbler is full of water than when it is empty?

10. What is meant by the limiting angle of total reflection?

11. A beam of light impinges as a perpendicular on one of the sides of a right-angled prism. It enters the prism and meets the hypothenuse. Prove that it will be there totally reflected.

12. Define the terms Principal Focus, Optical Centre, and Principal Axis of a Lens.

13. A double concave lens of focal length 1 inch is used to look at an object half an inch from the lens; where is the image?

14. With this lens, where will be the image of an object 1½ inches from the lens? Show how to find the focal length of a converging lens with a screen and a candle.

15. If the candle and screen be 8 feet apart, and the lens midway between them, what is the focal length?

16. It is desired to find the focal length of a diverging lens. All the lens is blackened, excepting a small circle at the centre of 3 ins. in diameter. A pencil of sunlight is made to pass through the aperture, and a screen is held behind the lens at a distance of 18 inches. It is then found that the circle illuminated is 1 foot in diameter. Show that the focal length is 6 inches.

17. In the above example what must be the distance of the screen from the lens that the diameter of the circle illuminated shall be twice that of the aperture of the lens?

18. With a converging lens the image of a candle is thrown on a screen which is 6 feet from the candle, and the focal length of the lens is 16 inches; find the distances of the candle and the screen from the lens.—*Ans.* 4 ft. and 2 ft.

19. With a double concave lens an observer, who knows his distance of most distinct vision to be 10 inches, sees an object most clearly when it is placed 15 inches from the lens; find the focal length of the lens.—*Ans.* 30 inches.

20. From § 45 prove that the focal lengths of a convex and concave lens have opposite signs.

21. Explain fully the meaning of the formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

22. From the formula prove that with a converging lens the image is real when at a greater distance than the principal focus, and virtual when nearer to the lens than the principal focus.

23. Show from the formula that with a lens which is thinnest in the middle the image is always virtual.

24. Find the focal length of a lens which throws the image of an object which is 5 feet distant on a screen which is 8 feet distant.

25. A double concave lens, having a focal length of 8 inches, is held at a distance of 2 inches from a small object; find the position of the image.

26. If in this case the object be ¼ of an inch in diameter, find the diameter of the image.

THE END.

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